



City Research Online

City, University of London Institutional Repository

Citation: Khalaf-Allah, M.A.E. (2007). Stochastic analysis of longevity and investment risk in the context of life annuities. (Unpublished Doctoral thesis, City University London)

This is the accepted version of the paper.

This version of the publication may differ from the final published version.

Permanent repository link: <https://openaccess.city.ac.uk/id/eprint/8554/>

Link to published version:

Copyright: City Research Online aims to make research outputs of City, University of London available to a wider audience. Copyright and Moral Rights remain with the author(s) and/or copyright holders. URLs from City Research Online may be freely distributed and linked to.

Reuse: Copies of full items can be used for personal research or study, educational, or not-for-profit purposes without prior permission or charge. Provided that the authors, title and full bibliographic details are credited, a hyperlink and/or URL is given for the original metadata page and the content is not changed in any way.

Stochastic analysis of longevity and investment risk in the context of life annuities

Marwa Abd El-Rahman Khalaf-Allah

A Thesis submitted for the degree of Doctor of Philosophy

**Faculty of Actuarial Science and Insurance
Sir John Cass Business School, City of London
The City University
London, England**

February 2007

Table of Contents

	Page
List of Figures	7
List of Tables	10
Acknowledgements and Declaration	12
Abstract	13
1 Introduction	14
2 Mortality projection methods and longevity risk in life annuities	21
2.1 Introduction	21
2.2 Mortality projection methods	23
2.2.1 Projection by extrapolation of mortality rates (The Logarithmic Method)	24
2.2.2 Mortality projections via reduction factors	26
2.2.3 Projections based on parametric models	30
2.2.4 Projections using relational models	34
2.2.5 Projection by cause of death	38
2.2.6 Lee-Carter projection model	40
2.2.7 The current practice of the CMIB and the use of penalised spline regression (P-splines)	42
2.3 Summary and Remarks	45

3	Analysing the distribution of life annuities using simulation techniques	47
3.1	Introduction	47
3.2	Mortality projection models of Sithole et al (2000)	49
3.3	Analysing Annuity values using simulation techniques	54
3.3.1	Methodology	55
3.3.2	Simulation results	57
3.3.2.1	Simulated distributions based on various mortality bases for female pensioners aged 60	57
3.3.2.2	Simulated distributions based on various mortality bases for male pensioners aged 60	61
3.4	Sensitivity Analysis	65
3.4.1	Changes in the distribution of annuity values with age	66
3.4.2	Changes in the distribution of annuity values with the rate of interest	73
3.4.3	Changes in the distribution of annuity values with level of Mortality assumed	78
3.5	Summary and Remarks	84
4	Entropy, Longevity and the Cost of Life Annuity	87
4.1	Introduction	87
4.2	Entropy and Mortality	89
4.2.1	Entropy for life expectancy at birth	89
4.2.2	Entropy for Annuities	93
4.3	Numerical results for H using English Life Tables	96
4.3.1	Numerical results for H over the whole age range	97

4.3.2	Numerical results for H at older ages	101
4.3.2.1	Numerical values for H for females and males – age 60	102
4.3.2.2	Numerical values for H for females and males – age 70	108
4.4	Numerical results for H using a mathematical model for mortality and allowing for mortality improvements	112
4.4.1	Gompertz law assumption	113
4.4.2	Mortality according to the (1991-1994) mortality table with mortality improvements allowed for using Sithole et al (2000)	120
4.5	Summary and Remarks	122
5	Bayesian Analysis for changes in the cost of life annuity due to longevity risk	125
5.1	Introduction	125
5.2	The Bayesian approach to Inference	127
5.2.1	Bayesian Inference, basis and advantages	127
5.2.2	Markov Chain Monte Carlo (MCMC) and Gibbs Sampling	128
5.3	Analyzing annuity values in a Bayesian framework using the Sithole et al (2000) mortality model	132
5.3.1	Methodology	132
5.3.2	Prior distributions and estimation of the parameters of Sithole et al (2000) mortality projection models	135
5.3.3	Analyzing annuity values	139
5.3.4	Convergence diagnostics	142
5.3.5	Measuring the effect of parameter uncertainty	152
5.4	Analyzing annuity values in a Bayesian framework using the Lee-Carter (1992) mortality projection model	153
5.4.1	Poisson log-bilinear model of Brouhns et al (2002)	155
5.4.2	Model and prior distributions	157

5.4.2.1	Likelihood function	157
5.4.2.2	Prior distributions for the parameters	157
5.4.3	Estimation of the Lee-Carter (1992) parameters	160
5.4.4	Analysis of annuity values	167
5.4.5	Convergence diagnosis	169
5.5	Summary and Remarks	175
6	Stochastic Bayesian analysis of the investment risk in a portfolio of life annuities	178
6.1	Introduction	178
6.2	Term structure of interest rate and Interest rate models	180
6.2.1	Definitions and Notation	181
6.2.2	Interest rate models	183
6.2.2.1	Short rate (spot rate) models	186
6.2.2.2	Forward rate models	189
6.2.2.3	Market models	191
6.3	Bayesian analysis of annuity values in a stochastic interest rate environment	192
6.3.1	The financial model	193
6.3.2	Methodology and prior distributions	196
6.3.3	Numerical results	197
6.3.4	Convergence diagnostics	201
6.4	Summary and Remarks	206

7 Conclusions	209
7.1 Summary	209
7.2 Areas of Further Research	215
Appendices	217
Appendix (1): Using Incomplete gamma function to find values of H – Mortality follows the Gompertz law	217
Appendix (2): Partial derivatives of the entropy measure (H)	221
Bibliography and References	224

List of Figures

	Page
3.1 Distribution of annuity payments for female pensioners aged 60 based on two mortality bases	60
3.2 Distribution of annuity payments for male pensioners aged 60 based on two mortality bases	64
3.3 Distribution of annuity payments for female pensioners based on two mortality bases for ages 70 and 80	70
3.4 Distribution of annuity payments for male pensioners based on two mortality bases for ages 70 and 80	71
3.5 Distributions of annuity payments for female and male pensioners aged 60 at different rates of interest	77
3.6 Distributions of annuity payments for female and male pensioners aged 60 for different scenarios of mortality improvements	83
4.1 Entropy values for different English life tables at different rates of interest for both females and males at age 0	100
4.2 Entropy values for different English life tables at different rates of interest for both females and males at age 60	106
4.3 Entropy values for different English life tables at different rates of interest for both females and males at age 70	111
4.4 Values for H when mortality follows the Gompertz law for different rate of interest and levels of mortality improvements	116
4.5 Values for H for different values of c and α - mortality follows the Gompertz law	118
4.6 Values for H at different rates of interest and mortality improvements allowed for using the Sithole et al (2000) model	121
5.1 Distributions of the parameter values of the Sithole model –male pensioners	138

5.2	Distribution of the parameter values of the Sithole model – female pensioners	139
5.3	Distribution of annuity values for male and female pensioners age 60	142
5.4	History plot for the parameters of the Sithole et al (2000) mortality projection model	145
5.5	Trace plot for the parameters of the Sithole et al (2000) mortality projection model	146
5.6	The Gelman-Rubin convergence statistic for the parameters of the Sithole et al (2000) mortality projection model	149
5.7	The autocorrelation function for the parameters of the Sithole et al (2000) mortality projection model	151
5.8	Estimates for α_x - different cases	162
5.9	Estimates for β_x - different cases	164
5.10	Estimates for κ_t - different cases	166
5.11	History plot for some selected parameters of the Lee-Carter(1992) mortality model	171
5.12	Trace plot for some selected parameters of the Lee-Carter(1992) mortality model	172
5.13	The Gelman-Rubin convergence statistic for some selected parameters of the Lee-Carter(1992) mortality model	174
6.1	Estimated values for zero coupon bond prices at time $t=1$ for different maturities T	198
6.2	Interest rates implied by zero coupon bond prices at different maturity time T	199
6.3	History plot for the parameters of the financial model and zero coupon bond prices ($P(1,T)$) at time $t=1$ for some selected maturities T	202
6.4	Trace plot for the parameters of the financial model and zero coupon bond prices ($P(1,T)$) at time $t=1$ for some selected maturities T	203

6.5	Gelman-Rubin convergence statistic for the parameters of the financial model and zero coupon bond prices $(P(1,T))$ at time $t=1$ for some selected maturities T	204
6.6	Autocorrelation function for the parameters of the financial model and zero coupon bond prices $(P(1,T))$ at time $t=1$ for some selected maturities T	206
A2.1	The partial derivative of H with respect to α at different interest rates	223

List of Tables

	Page
3.1 Sithole et al (2000) parameters' estimates for life office pensioners 6-parameters models	52
3.2 Sithole et al (2000) reduction factors	54
3.3 Summary of the descriptive statistics of the simulated distributions- ferr pensioners aged 60	58
3.4 Summary of the descriptive statistics of the simulated distributions- male pensioners aged 60	62
3.5 Summary of the descriptive statistics of the simulated distributions for different ages at inception	67
3.6 Summary of the descriptive statistics of the simulated distributions for different rates of interest	75
3.7 Summary of the descriptive statistics of the simulated distributions for different rates of interest	81
4.1 Values for the complete life expectancy and the entropy measure for males and females, United States	92
4.2 Entropy values over the whole age range – Females	98
4.3 Entropy values over the whole age range – Males	99
4.4 Entropy values for Females - age 60	103
4.5 Entropy values for Males - age 60	104
4.6 Entropy values for Females - age 70	108
4.7 Entropy values for Males - age 70	109
5.1 Summary of the descriptive statistics of the parameters of Sithole et al (2000) -male and female life office pensioners	137

5.2	Summary of the descriptive statistics of the simulated distributions of annuity payments for male and female life office pensioners	141
5.3	Estimation error of the mean value of the simulated distribution of annuity payments	152
5.4	Summary of the descriptive statistics of the simulated distributions of annuity payments for female life office pensioners using the Lee-Carter(1992) projection model	168
6.1	Summary of the descriptive statistics of the simulated distributions of annuity payments for female life office pensioners	200

Acknowledgements

“My success can only come from God. In Him I trust, and unto Him I look.”

(Surat Hud :87)

It takes more than words to express my gratitude and heartfelt thanks to all those who stood beside me all through, for all the deeds that they have done for which I can never repay.

I start with dedicating this dissertation to the memory of my dear professors Dr. Fathy Ibrahim and Dr. Mohamed Tawfeek El-Mansoury to whom no words can ever be enough and whom I will never forget. Just knowing them, has shaped my life more than I could ever tell.

Sincere thanks to my supervisors, Professor Steve Haberman and Professor Richard Verrall for their dedication, advice and granted guidance.

It is through our friends we truly begin to know ourselves and it is their gift of friendship that makes life worthwhile! Thanks to all my true friends.

A very special thank you and a very special place in my heart to my two best friends, Laxmi Remer (Iyer) and Reem Mamdouh for always being there for me in every way and for always believing in me.

Finally, yet the most important, I would like to thank my parents for their unconditional love and support and for preparing me to be the person that I am today. I would like also to express my deep appreciation to my dear brothers Mohamed and Mahmoud (Ketket) for their invaluable support.

Declaration

I grant powers of discretion to the University Librarian to allow the thesis to be copied in whole or in part without further reference to the author. This permission covers only single copies made for study purposes, subject to normal conditions of acknowledgment.

Abstract

This thesis aims to investigate the effect of longevity risk in the context of life annuities. It develops different tools and frameworks to measure this risk as a step to facilitate the risk management of longevity risk. Particular attention is directed to stochastic modelling which allows the uncertainty of future projections to be incorporated. Hence, simulation methods are used to consider the distribution of the annuity cost, as well as the more often quoted point estimates.

A theoretical extension of the use of the entropy measure applied in population biology by Demetrius (1976) has been developed to measure the effect of a proportionate change in the force of mortality on the cost of life annuity. The properties of the corresponding entropy measure have been then investigated using the Gompertz and the Sithole et al (2000) mortality projection models. Numerical results suggest that, at very high or low levels of mortality, the effect of mortality changes on the value of life annuity is of reduced importance. A full Bayesian model has been developed which incorporates the estimation of the parameters of both the Sithole et al (2000) and the Lee - Carter (1992) mortality projection models within the simulation of the annuity cost. This has been extended to an environment in which the future rates of interest are stochastic. The effect of parameter uncertainty of the Sithole et al (2000) mortality projection model has been considered and shown to be less important than the associated model uncertainty.

Chapter 1

Introduction

Mortality rates in developed countries have been persistently declining over time and are expected to continue to decline in particular at adult and older ages in many of these countries. However, given the irregularity of the rate of decline over time there is much debate regarding the most suitable method of forecasting the future improvement in mortality.

Further aspects of mortality trends can be captured by looking at the survival function curve. As noted by Olivieri (2001) and Pitacco (2004), mortality experience over the last decades shows some aspects affecting the shape of the curves representing the mortality as a function of the attained age. In particular, the following features have been noted:

- An increasing concentration of deaths around the mode (at old ages) of the curve of deaths. This phenomena is known as rectangularization as the survival curve moves towards a rectangular shape.
- Expansion of the survival function, where the mode of the curve of deaths moves towards very old ages.

- More recently, a further aspect has been observed; higher levels and larger dispersion of deaths at young ages (known as the young mortality hump).

The above mentioned mortality trends clearly affect claim frequencies in life insurance. In particular, the first two aspects affect living benefits, whilst the last aspect affects death benefits. Rectangularization, the expansion phenomena and decreasing mortality profiles have been experienced in a number of countries: for a recent international comparison, refer to MacDonald et al (1998).

Annuities are specially designed to meet a financial need for providing an income for the remainder of the life of the insured. In this case, we can see that the main risk that the providers of these contracts face is longevity risk. In the context of life annuities, longevity risk can be defined as the variation in actual financial experience from that expected, which is attributable to the actual mortality experience being lighter than that which has been assumed. With the likelihood of further reductions in future mortality rates and given the recent trends mentioned above, incorporating mortality improvements is essential for annuities as the cost of an annuity is very sensitive to the level of mortality assumed.

It is noteworthy that, as pointed by Pitacco (2004), mortality trends and relevant effects on life annuities were clearly perceived at the beginning of the 20th century. For instance, Nordenmark (1906) has pointed out that improvements in mortality must be carefully considered when pricing a life annuity and that in particular cohort mortality should be addressed to avoid underestimation of future liabilities.

This thesis aims to investigate the effect of mortality improvements on the expected costs of annuities. This investigation is performed as an attempt to answer questions such as: given the improvement in future mortality rates, what is the effect of initial ages, gender, rate of interest and the level of mortality improvement on the additional cost implied?. To this end, simulation methods are used to consider the distribution of the annuity cost, as well as the more often quoted point estimates. Then the entropy measure applied in population biology by Demetrius (1976) is extended to measure the effect of any changes in the force of mortality on the cost of life annuity for different interest rate scenarios and levels of mortality improvements. This is one of the main contributions of this thesis. The other main contribution is that we then develop a Bayesian model which incorporates the estimation of the parameters of the mortality projection model within the simulation of the annuity cost. This allows us to consider the effect of parameter uncertainty on the projected distribution of the annuity cost. This Bayesian model has been then extended to an environment in which the

future rates of interest are stochastic which will allow an integrated analysis of the effect of demographic and financial risks and their interaction on the cost of life annuities.

The thesis is organised in seven chapters, which are briefly introduced below.

Chapter 2 – Mortality projection methods and longevity risk in life annuity.

Chapter 2 discusses the recent trends in mortality in particular mortality improvements over the past century and the corresponding effect on the cost of survival benefits hence the need arises to project future mortality and accurately allow for any further improvements. There are many approaches to forecasting mortality rates that are described in the actuarial literature. In chapter 2, we present some of the standard extrapolative approaches used in the actuarial literature. These represent the most commonly used forecasting models in actuarial applications.

Chapter 3 – Analysing the distribution of life annuities using simulation techniques. In this chapter the effect of mortality improvement on the cost of life annuity is investigated using simulation techniques and scenario analysis. Simulations techniques are used to model a particular path that a group of persons may follow during their life time by allowing the time of death for each person to be a random variable. Using this we can investigate the properties of

the distribution of outcomes. Scenario analysis has been used to investigate the effect of age at inception, gender, assumed interest rate and the level of mortality assumed on the distribution of annuity payments. The results are illustrated using UK life office pensioners data from the Continuous Mortality Investigation Bureau and mortality improvements have been allowed for using the Sithole et al (2000) mortality projection model.

Chapter 4 – Entropy, Longevity and the cost of life annuity. Chapter 4 extends the theory of the entropy measure applied in population biology by Demetrius (1976) to measure the effect of a proportionate change in the force of mortality on the cost of a life annuity for different interest rate scenarios and levels of mortality improvements. This allows different sources of risk in a life annuity contract to be summarized in a one figure index. Numerical values for the entropy measure are derived using an approach which extends that of Keyfitz (1977). Results are illustrated using English life tables over the period from 1851 to 1991 and also by applying different mathematical models for mortality projections such as the Gompertz and the Sithole et al (2000) mortality projection models for both males and females aged 60 as an attempt to get a better understanding regarding the properties of the entropy measure. This is followed by testing the sensitivity of the results obtained with regard to the different factors that are likely to affect the value of the entropy measure. In this

study the effect of gender, age, assumed interest rate and the level of mortality improvement are investigated.

Chapter 5 – Bayesian analysis of the changes in the cost of life annuity due to longevity risk. In Chapter 5 a full Bayesian model has been constructed to investigate longevity risk and the effect of mortality improvements on the cost of a life annuity. Simulated distributions of annuity payments are obtained in a manner similar to that in chapter 3 except that the Bayesian approach combines the estimation of the parameters of the mortality projection models together with the simulation of the annuity cost. The chapter starts by introducing the Bayesian approach to inference and Markov chain Monte Carlo (MCMC) method. Then a full Bayesian model is constructed to implement the corresponding MCMC-Bayesian analysis needed to estimate the parameters of both the Sithole et al (2000) and the Lee-Carter (1992) mortality projection models, and hence obtain the corresponding simulated distributions of annuity payments. For the Sithole et al (2000) model, results are illustrated using UK life office pensioners data for both males and females aged 60, 70 and 80. The results have then been compared to the corresponding ones in chapter 3. Also, three different data sets have been used to perform the analysis using the Lee-Carter (1992) model. These are: England and Wales male mortality experience, 1950-1998 inclusive (Case A), UK mortality experience, 1961-2003 inclusive (Case B) and CMI data for female life office pensioners, 1983-1996 inclusive (Case C). An approach for

measuring the effect of parameter uncertainty is then presented and implemented.

Chapter 6 – Stochastic Bayesian analysis of the investment risk in a portfolio of life annuity. Chapter 6 extends the analysis of the cost of annuity to an environment in which the future rates of interest are stochastic. This allows an integrated analysis of demographic and financial risks and their interaction. As in chapter 5, this analysis is performed in a Bayesian framework in which the parameters of both the financial model and the mortality projection models are estimated. The chapter starts by introducing the term structure of interest rates. This is followed by presenting some of the standard interest rate models in the literature, and then a full Bayesian model is constructed to estimate any involved parameters and simulate the corresponding distributions of the cost of annuity. Mortality improvements are allowed for using the Lee-Carter (1992) mortality projection model and the interest rate model used is the one developed by Ballotta and Haberman (2003) which makes use of the one factor Heath-Jarrow-Morton framework for the term structure of interest rate. The analysis has been performed for female life office pensioners aged 60,70 and 80.

Chapter 7 – Conclusions. The last chapter summarises the main findings and the conclusions in each of the previous chapters. Furthermore, a discussion of possible extensions and further work are presented.

Chapter 2

Mortality projection methods and longevity risk in life annuities

2.1 Introduction

Over the past century mortality rates in the developed countries, including the UK, have improved remarkably (Charlton, 1997). There is still much uncertainty as to the processes that cause ageing and there is much debate as to whether there are upper limits to human longevity. However, it is clear that there is no conclusive evidence to suggest that current mortality rates are close to reaching any kind of lower bound (see Thatcher, 1999). This persistent decrease in mortality rates has become a major concern of annuity and pension providers. This is particularly true for mortality improvements for post-retirement ages which have a significant financial impact as far as the cost of survival benefits is concerned, as a small percentage difference in mortality at these ages can translate into substantial extra cost. Under these conditions of improving mortality, the projection of future annuitants' and pensioners' mortality is essential. To ignore improvements would be to endanger the financial stability of the insurer selling policies providing such survival benefits or defined benefit pensions schemes providing retirement benefits, taking into consideration that

longevity has a direct impact on the cost of survival benefits for both annuitants and pensioners.

The effect of longevity risk (Olivieri and Pitacco, (2002)) is even more important with the combination of improving mortality and falling interest rates, which have shaken the annuity market in recent years, especially for products offering guarantees (Ballotta and Haberman, (2003)). The calculation of expected present values thus requires an appropriate mortality projection in order to avoid underestimation of future costs.

There are a number of broad approaches to forecasting mortality rates – using models based on the underlying biomedical process, causal models based on econometric-type relationships, and trend models that are extrapolative in character. We will consider only the last category in this chapter.

Projecting the behavior of future mortality rates is a rather complicated process given that there are many factors that are likely to affect future mortality rates and that the effect of some of these (for example, social, economic, cultural and ethnic factors) maybe difficult to measure or even to model. These factors affect different people differently, which makes the process of forecasting the future course of mortality change a challenge. However, due to the importance of mortality projection as mentioned above, many attempts have been made and

many methods have been proposed for projecting mortality in the future. In general there is no single best method and the choice of the appropriate method will depend on the purpose of the projection and the quality and the quantity of the data available.

This chapter is organized as follows: In section 2.2 various methods of projecting mortality are discussed. Section 2.3 will address some remarks.

2.2 Mortality projection methods

A number of projection models are described in the actuarial literature. The standard extrapolative approaches used in the literature include but not limited to: a) models based on the independent projection of age-specific mortality rates or forces of mortality, including mortality reduction factor models; b) relational models based on the logit transformation; c) models based on graduating mortality rates with respect to age for specific time period and then projecting the parameters; d) models based on graduating mortality rates with respect to age and time simultaneously; e) the Lee-Carter method. Description of these methods can be found in Lee and Carter (1992), Benjamin and Soliman (1993), Renshaw et al (1996) and Pitacco (2004). Further, Carmer and Wold (1935), gives an interesting historical reference of the early attempts to consider mortality trends together with presentations and discussions of the early projection models.

In this section, some of the standard extrapolative approaches used in the actuarial literature are presented. These represent the most commonly used forecasting models in actuarial applications.

2.2.1 Projection by extrapolation of mortality rates (The Logarithmic Method)

Projection by extrapolation of the mortality rates is considered the simplest method of projecting age-specific mortality rates and the most widely used. This method is based on the assumption that the proportionate reduction from one year to another in the age-specific death rate is relatively constant over a fairly extensive but not indefinite time period. This means that, if the mortality rate at age x is plotted for successive years, the curve would be close to a straight line. Usually a logarithmic transformation of the mortality rates is used to improve linearity, and the projection is then performed by plotting the logs of mortality rates over time and fitting a straight line to the data sequences.

However, as the logarithmic decline of the mortality rate does not remain the same for indefinite periods of time but tends to change, it is therefore necessary to examine the extent of the period over which the decline seems to be relatively constant.

Extrapolation may be performed either graphically or by mathematical formula. Under the graphical approach, mortality rates at each selected age x at recent time periods t are plotted against t . A smooth curve is drawn through the points and similar curves are drawn on the same graph for neighboring ages x . The curves are all then extrapolated to yield projected values of $q_{x,t}$. Projected mortality rates at intervening ages are found by interpolation or by multiplying the base mortality rates (initial mortality rates) at those ages by the projected reduction factor at the nearest selected age.

The most commonly used formula for mathematical extrapolation is:

$$q_{x,t} = \beta_x \gamma_x^t, \quad (2.1)$$

where $q_{x,t}$ is the mortality rate at age x experienced in year (or time period) t , β_x is the level of mortality at age x at a particular point in time, that is, the initial level of mortality and γ_x , ($0 < \gamma_x < 1$) is the annual rate of improvement in mortality at age x .

The equivalent formula for the logarithmic transformation of $q_{x,t}$ is

$$\ln q_{x,t} = B_x + tC_x \quad (2.2)$$

where $B_x = \ln \beta_x$ and $C_x = \ln \gamma_x$.

Formula (2.1) allows the mortality rate at age x to decrease indefinitely towards zero. An alternative formula is preferred if an ultimate level of mortality at age x , α_x , is assumed:

$$q_{x,t} = \alpha_x + \beta_x \gamma'_x. \quad (2.3)$$

This projection procedure can also be applied to quantities such as the central rates of mortality m_x , the forces of mortality μ_x and the mortality odds $q_x/(1 - q_x)$.

Once the parameters of the mathematical formula have been estimated, the projected mortality rates are then calculated using formula (2.1) or (2.3) as appropriate.

2.2.2 Mortality projections via reduction factors

Reduction factors provide a tool that is widely used in actuarial practice for projecting mortality. For example, in the UK, the Continuous Mortality Investigation Bureau (CMIB) preferred approach to projecting mortality is via reduction factors.

UK life offices have co-operated in the collection of mortality statistics for almost 180 years. The CMIB has collected data from the majority of life offices since the 1920s, and from time to time has published graduated mortality tables for assured lives, annuitants and pensioners. In general, the Committee has considered it essential, when publishing the new tables for pensioners and immediate annuitants, to publish projection factors in order to allow for

improvements in mortality with the passage of time because as noted earlier, improving longevity has a direct upwards impact on the cost of survival benefits for both annuitants and pensioners.

The methodology proposed by the CMIB for projecting the future improvements in the mortality of both Pensioners and Annuitants which has been used since the preparation of the “80” series – is as follows. For a given investigation period, data are graduated by fitting a “Gompertz-Makeham” class of formula:

$$\mu_x = GM_x(r,s) = \sum_{i=1}^{r-1} \alpha_i x^i + \exp \left[\sum_{j=0}^{s-1} \beta_j x^j \right] \quad (2.4)$$

When $r=0$ and $s=2$, this expression equates with Gompert’s law, While when $r=1$ and $s=2$ a Makeham equation is given. When $r=0$ the polynomial term is absent and when $s=0$ the exponential term is absent (Forfar et al 1988).

The force of mortality at age x is given by, $\mu_x = GM(r,s)$. Then the projected rates are produced by applying time reduction factors to the tables resulting from the graduation (also known as the base tables).

So for a life attaining age x after t years from the base year, the formula for the projected mortality rate at time t will be as follows:

$$q_{x,t} = q_{x,0} \cdot RF(x,t) \quad (2.5)$$

where

$q_{x,0}$ is the value of q_x in the relevant new 'base' table.

$RF(x,t)$ is the "reduction factor" for age x and time t .

For the 1991-1994 mortality tables, the CMIB has proposed a mortality improvement model for pensioners and annuitants, under which the rate of mortality at each age is assumed to decrease exponentially to an age specific limiting value, with the speed of convergence to the limit depending on age.

The reduction factor model recommended by the CMIB for a life attaining age x after t years from the base year for all experiences is:

$$RF(x,t) = \alpha(x) + [1 - \alpha(x)] \cdot [1 - f(x)]^{\frac{t}{20}} \quad (2.6)$$

with

$$\alpha(x) = \begin{cases} c & x < 60 \\ 1 + (1-c) \cdot \frac{(x-110)}{50} & 60 \leq x \leq 110 \\ 1 & x > 110 \end{cases}$$

and

$$f(x) = \begin{cases} p & x < 60 \\ \frac{(110-x) \cdot p + (x-60) \cdot q}{50} & 60 \leq x \leq 110 \\ q & x > 110 \end{cases}$$

where

$$c = 0.13, p = 0.55, \text{ and } q = 0.29.$$

This mortality improvement model assumes that below age 60 the improvement in mortality depends only on the time in years from the origin (t), at ages between 60 and 110 the improvement in mortality is assumed to depend on both the age (x) and the time (t), while for ages 110 and above no improvement is assumed.

Also no sex differential is assumed under this model, i.e. the improvement in mortality is considered to be the same for both males and females.

We note that, the resulting reduction factors are constructed without a specific modelling structure (see Renshaw and Haberman (2003a)), and that the calculation of the reduction factors is based on a 20-year time span.

A similar approach has been proposed by the Society of Actuaries in the USA to calculate the projected probabilities of death $q_x(y)$ using the 1994 mortality rates as the base table and the improvement factors AA_x as follows:

$$q_x(y) = q_x(1994)(1 - AA_x)^{y-1994} \quad (2.7)$$

Mortality projection via reduction factors represents a practical and convenient approach to mortality forecast, which has resulted in recent contributions to the modelling of reduction factors: see Renshaw and Haberman (2000) and Renshaw and Haberman (2003a). In the later paper, the Lee-Carter projection method (will be discussed later in section 2.3.6 of this chapter) is re interpreted within the context of mortality reduction factors.

2.2.3 Projections based on parametric models

This approach involves fitting a mathematical curve to the age progression of death rates over the whole life span. Many attempts have been proposed to find a mathematical function that can be considered as a law of mortality. One of the earliest attempts was that of Gompertz (1825) who argued on physiological grounds that the intensity of mortality increases in equal proportions in equal intervals of age, leading to an exponentially increasing force of mortality. viz:

$$\mu_x = Bc^x \quad (2.8)$$

where B, c are constants and $B > 0, c > 1$.

Later on a development of Gompertz formula was made by Makeham (1860) by introducing a constant component (A) as well as the exponentially increasing

component for the force of mortality, as a reflection of the division of causes of death into two types, those due to chance and those due to deterioration, giving:

$$\mu_x = A + Bc^x \quad (2.9)$$

The use of Makeham based projected survival models has been utilised by Cramer and Wold (1935) to graduate and extrapolate Swedish mortality rates from 1801 to 1930 for lives aged between 30 and 90.

However, it has soon been found to be difficult to obtain a satisfactory representation of the whole life span by using such a simple formula, and more complicated formulas have therefore been developed. For example, Thiele (1872) proposed the following formula:

$$\mu_x = a_1 e^{-b_1 x} + a_2 e^{-\frac{1}{2} b_2 (x-c)^2} + a_3 e^{b_3 x} \quad (2.10)$$

where the first term is a decreasing Gompertz curve representing childhood mortality, the last term is a Gompertz curve representing old age mortality and the middle term is a normal curve representing mortality in adulthood. Perks (1932) introduced a new family of curves of the general form:

$$\mu_x = (A + Bc^x)/(Kc^{-x} + 1 + Dc^x) \quad (2.11)$$

Perks rationalised this procedure by finding an analogy between Gompertz law and the physical concept of entropy change. A review of Perks' curves can be found in Benjamin and Pollard (1980).

More recently, the following formula proposed by Heligman and Pollard (1980) has produced good results over the whole age span:

$$\frac{q_x}{p_x} = A^{(x+B)^c} + D \exp\{-E(\ln x - \ln F)^2\} + GH^x \quad (2.12)$$

The Heligman-Pollard model has eight parameters denoted by the letters A to H that need to be estimated from each observational data. This model is made up of three terms, each of which represents mortality behaviour for a specific stage of the life span: infancy, young adulthood and a Gompertz curve representing mortality at older ages. Hence, when considering mortality at the older ages only, the first two terms can be neglected so that the Heligman and Pollard law is very similar to the Gompertz law at the older ages (see Thatcher (1990), Congdon (1993)). Despite the complicated formulation of the model and the number of parameters involved, this model has the important advantage of representing specific mortality patterns for different stages of life span with a single equation.

The Heligman-Pollard law, can be generalised in many ways, for example, Heligman and Pollard (1980) proposed the so-called Heligman-Pollard second law by changing the third term as follows:

$$\frac{q_x}{p_x} = A^{(x+B)^r} + D \exp\left\{-E(\ln x - \ln F)^2\right\} + \frac{GH^x}{1 + KGH^x} \quad (2.13)$$

Forfar and Smith (1988), have fitted the Heligman-Pollard curve to the graduated rates from English life tables (ELT) 1-13, for ages 0-85, for both males and females using both equation (2.12) and (2.13).

Recently, Sithole et al (2000) have projected mortality rates for UK immediate annuitants and life office pensioners using generalized linear models. The Sithole et al (2000) projection models have been developed by fitting the Renshaw et al (1996) mortality model and represent an extension of the Gompertz-Makeham class of formulas as in equation (2.4). The Sithole et al (2000) projection models will be discussed in detail in chapter 3 of this study.

In general, once a suitable formula for the experience under question is developed, it can be used for projection purposes as follows. Trends in the parameters are extrapolated to provide estimates of the parameters at future time

periods t . Projected age-specific mortality rates are then obtained by substituting the projected parameters and the various ages into the relevant formula.

2.2.3 Projections using relational models

Relational models involve relating life table measures with those from a standard life table. A relational approach can be particularly useful in cases where there are not enough detailed or reliable data to construct life tables.

One way of using relational projection models can be by reference to model life tables, this approach can be implemented as follows. Firstly, a set of model life tables is chosen which, is believed, to represent and will continue to represent the mortality of the population of interest. The set of model tables may involve a single parameter or two or more parameters. In the single parameter case, the parameter of the system is measured in the population at each of several time periods. Any trend in the parameter is extrapolated graphically or by a statistical method to provide estimates of the parameter at future time periods. Projected age-specific mortality rates are obtained by entering the model life table system for the various projected values of the parameter.

The first set of model tables was constructed by the United Nations in 1955.

There are a number of model life tables that have been developed over the past

40 years, such as Coale and Demeny, 1966; Brass, 1971 and the Organization for Economic Corporation and Development, 1980.

Another form of relational projection is projection by reference to a more advanced population, this can be done by considering a more advanced population with adequate mortality statistics, and with a mortality history that is similar to the population under study. The mortality characteristics of the population under study are then compared with those of the more advanced population and similarities are noted. For example, it may be that the mortality of the population under study is essentially the same as that of the more advanced population but with a time lag of some years which appears to be slowly shortening. Projections of mortality for the population under study are taken as those mortality rates already experienced by the more advanced population and (when necessary) projected for the more advanced population.

Relational projections can also be performed by reference to an “Optimal” life table that is attainable under ideal conditions: Many people have addressed the question: “What is the optimal life table one could expect in respect of a given population?” and a variety of approaches have been adopted in an attempt to answer the question. The idea of an “Optimal” table was proposed by Bourgeois-Pichat (1952) asking a similar question: “Can mortality decline indefinitely or there is a limit, and if so, what is this limit?”. Determining a limiting table

requires a number of assumptions about the trend in various mortality causes, so that an analysis of mortality by the causes of death is required. Benjamin (1982) has made some 'extreme assumptions' about improvements in mortality by cause in an attempt to come up with a life table under optimal conditions. Using these basis and using England and Wales data, he estimated an ultimate expectation of life at birth of 81.3 for males and 87.1 for females. This approach can be used as follows: a suitable optimal life table attainable under ideal conditions is selected from those developed by other researchers or developed from the population's own cause-of-death data as done by Benjamin (1982), then a decision is taken as to how the population will approach the optimal mortality schedule and over how long it will do so.

A method of forecasting mortality that utilizes the relational method has been proposed by Brass (1971) and (1974). In order to express relationships among mortality of various populations, Brass (1971) focuses on the logit transformation of the survival function, namely

$$\Lambda_{x,t} = \frac{1}{2} \log \left(\frac{l_{0,t} - l_{x,t}}{l_{x,t}} \right) \quad (2.14)$$

Brass notes empirically that $\Lambda_{x,t}$ can be expressed in terms of the logit of a standard population, Λ'_x , via a linear relation of the form:

$$\Lambda_{x,t} = \alpha_t + \beta_t \Lambda_x^s \quad (2.15)$$

The parameter α_t reflects the level of mortality while the parameter β_t indicates the relationship of mortality across the different ages relative to the standard.

The values of the parameters α_t and β_t are almost independent of the age x . So the problem of projecting mortality reduces to the problem of extrapolating the two series α_t and β_t , and then the projected values of the survival function can be derived from the inverse logit transformation.

The assumption of linearity between the logit curves used in (2.15) does not always exist, Brass (1971) suggests that linearity in some cases can be obtained by using the differences between the logits for different time periods and the logit taken as the standard. Brass has used this system to obtain results for Sweden as well as England and Wales using generation data (Benjamin and Soliman (1993)).

Congdon (1993) has applied the relational approach for projection purposes by using male life tables for greater London over 1971–90. Equation (2.15) is fitted for each life table. The series of values for the parameters α_t and β_t are then modeled by time series ARIMA methods.

2.2.5 Projection by cause of death

Future mortality can be projected by extrapolating mortality by the various causes of death separately. This involves a considerable amount of work, but it can reduce the chance of certain errors in the projection of overall rates. This approach was first used by Pollard (1949) to project Australian national mortality. Pollard distinguished 13 cause groups: influenza, pulmonary tuberculosis, epilepsy, bronchitis and pneumonia, accidents, growths, intercranial lesions, diabetes, nephritis, appendicitis, diseases of the circulatory system, ulcers of the stomach and duodenum, and other causes. Mortality rates for these causes were calculated for selected ages for each of the years 1921-1938 and projected forward graphically towards 1970. In the event, the projected age-specific mortality rates obtained by combining the projected age-specific rates by cause were significantly higher than the rates obtained by any other method. Pollard (1987) argues that the high mortality rate predicted reflects the rapidly rising mortality rates from circulatory system diseases and, to a lesser extent, accidents.

Mortality can be projected by cause of death as follows. First, cause-of-death statistics are used to calculate age-specific mortality rates by cause at each of several recent time periods for selected ages. Then, the age-specific mortality rates by cause are projected separately for the selected ages. The projected age-

specific mortality rates by cause are then combined to yield the projected mortality rates at the selected ages. Projected mortality rates at the intervening ages are found by interpolation.

Projection by cause of death has its own attractiveness as a method as it offers a useful insight into the changing incidence of different causes of death. However, it is a complex approach to apply, as the correlation between different causes has to be allowed for. For example, heart diseases and lung cancer are correlated as both are linked to smoking habits. Carriere (1994) investigated the effect of removing heart and cerebrovascular diseases as a cause of death from the U.S. population. Assuming that these diseases were dependent on other causes, the dependence was modelled using the theory of copula functions. For a more precise definition refer to Carriere (1994) or Schweizer and Sklar (1983).

It also has an inherent tendency to produce lower future improvement rates than experienced in the past. This is because the causes showing the greatest past improvements will become relatively less common causes of death as time passes.

A further problem is the difficulty in identification of the cause of death for elderly people. Detailed cause of death records can be unreliable for the elderly,

largely because of misclassification. This means that the data will be of doubtful quality at those ages.

2.2.6 Lee-Carter projection model

In 1992 Lee and Carter proposed a simple model for describing the change in mortality as a function of a single time index. The model represents the log of the age specific central rate of mortality as the sum of two terms; an age-specific term that is independent of time and a second term which is the product of a time varying parameter reflecting the general level of mortality, and an age-specific component that represents the extent to which the mortality rate at each age will vary due to changes in the general level of mortality. The model is based on the equation:

$$\log m_{x,t} = \alpha_x + \beta_x k_t + \varepsilon_{x,t} \quad (2.16)$$

where

$m_{x,t}$ is the central death rate of mortality for age x at time t ;

α_x describes the average shape of the age profile over time;

β_x describes the pattern of deviations from the age profile when the parameter k_t varies;

k_t describes the variation in the rates of death with time t ; and

$\varepsilon_{x,t}$ is an error term with mean 0 and variance σ_ε^2 .

The model cannot be fitted by a simple regression approach, since there is no observable variable on the right hand side of (2.16). Moreover, it allows for several solutions, and, to eliminate this identifiability problem, Lee and Carter suggest that β_x and κ_t be normalized by imposing the following constraints;

$\sum_x \beta_x = 1$; and $\sum_t \kappa_t = 0$, which in turns forces each α_x to be an average of the

log central death rates over the calendar years in the data set.

Model fitting proceeds as follows:

- 1- Estimate α_x as $\hat{\alpha}_x = \log \left\{ \prod_{t=1}^n \hat{m}_{xt}^{1/n} \right\}$, the logarithm of the geometric mean of the crude mortality rates, averaged over all t, for each age.
- 2- Compute the matrix of statistics $(z_{xt}) = (\log(\hat{m}_{xt}) - \hat{\alpha}_x)$ and then estimate κ_t and β_x as the respective first right and first left singular vectors in the SVD of the matrix (z_{xt}) subject to the above constraints.

Taking the estimated values for α_x and β_x , the estimated κ_t will be adjusted so that the actual number of deaths is equal to the total expected number of deaths. In order to proceed to mortality projection, the time factor κ_t is modelled as a stochastic time series process using standard Box-Jenkins procedures. Lee

(2000) observes that in most applications of the method so far, κ_t is modelled by an ARIMA (0, 1, 0) process, or random walk with drift. Hence the forecast of each age-specific mortality rate s periods ahead from base period t_0 can be obtained using the following equation:

$$\log(m_{x,t_0+s}) = \hat{\alpha}_x + \hat{\beta}_x \hat{k}_{t_0+s} \quad (2.17)$$

where the $\hat{\cdot}$ indicates estimates of the associated parameters.

Brouhns et al (2002a) and Brouhns et al (2002b) investigate possible improvements to the Lee-Carter method whereby the number of deaths are modeled as Poisson random variable. This extension of the Lee-Carter will be considered in details later in Chapter 5.

2.2.7 The current practice of the CMIB and the use of penalised spline regression (P-splines)

The most recent set of mortality projections (these for use with the "00" series tables) were presented by the CMIB in working paper 1 (CMI 2002). Since the 'a(55)' tables were published in 1953, the CMIB has customarily provided mortality projections when it has published mortality tables. The new projections were distinctive in three important aspects: Firstly, the projections allowed for the cohort effect, which is defined as the dependence of mortality improvement

rates on a person's year of birth. The cohort effect proved to be important in analyzing the UK mortality experience (see Willets (2004)). Secondly, these projections were extrapolations based on results of a methodology new to actuaries, namely penalised spline regression (P-splines). Lastly, three alternative projections were offered, instead of the traditional single projection. The three scenarios were called 'short', 'medium' and 'long' cohorts, differing in the length of time over which the cohort effect was assumed to persist. These periods were chosen arbitrarily and no probabilistic interpretation was possible.

In working paper 3 (CMI March 2004), projections were contrasted based on time series models and regression models. Time series models were exemplified by the Lee-Carter mortality model (see section 2.2.6), and regression models by the penalised spline (P-spline). A brief description of this methodology is given below.

The P-Spline model

Consider the number of deaths at age x in calendar year t ($D_{x,t}$) to have a Poisson distribution with mean ($E_{x,t}\mu_{x,t}$), where $E_{x,t}$ is the corresponding exposure to risk and $\mu_{x,t}$ is the corresponding force of mortality. Using regression approach, the relationship between $\mu_{x,t}$ and the variables x and t can

be expressed through the choice of some basis functions

$b_1(x,t), b_2(x,t), \dots, b_n(x,t)$, such that:

$$\mu_{x,t} = a_1 b_1(x,t) + a_2 b_2(x,t) + \dots + a_n b_n(x,t) \quad (2.18)$$

The question is, how can we choose a suitable set of basis functions $b_i(x,t)$, and what criterion do we apply to choose the best fitting regression coefficients a_i . Splines represent one answer to these questions as they provide an alternative choice of basis functions. Splines have been used before in actuarial practice in the UK to graduate the last few English life tables (see Benjamin and Pollard (1980)). A spline of degree m is simply a curve made up of segments of polynomials of degree m , such that where the segments join, their derivatives up to order $m-1$ are equal. The term 'B-splines' is often used to denote a set of basis spline. A B-spline graduation require us to reach a balance between goodness of fit (achieved by adding more and more splines to the basis) and smoothness (achieved by limiting the number of splines in the basis), another approach is penalised spline, or P-spline, which makes no attempt to keep the number of basis splines small instead we make sure that the basis is rich enough to provide a good fit. Then we impose an explicit penalty on lack of smoothness, represented conveniently by lack of smoothness in the progression of the coefficients a_i . Then the precise number of basis functions almost ceases to

matter; the trade off between smoothness and the goodness of fit is achieved by choosing a large penalty (prefer smoothness) or small penalty (prefer goodness of fit). For more details on the use of the P- spline approach to mortality projection refer to Currie, Durban & Eilers (2004) and CMI working papers 1, 3, 15 and 20.

2.3 Summary and Remarks

There are many approaches and models that have been proposed to project mortality rates. The choice of the method to be used will depend on the data and their reliability, the resources available for the project and the purpose for which the projection is required. In general, no mortality projection basis can ever be correct. Also it is very important to bear in mind that many of the projection methods discussed in this chapter suffer from drawbacks. For example, and as noted by Brouhns et al (2002a) the use of a law-based approach to mortality forecast might not be appropriate if we consider the strong dependence between the estimated parameters, and hence, univariate extrapolation of the parameters might be misleading, and while a multivariate time series model for the parameters is possible it might lead to computational intractability.

While the methodology suggested by Lee-Carter (1992) avoids these problems, it implicitly assumes – by using the ordinary least square method to estimate the

parameters- that the errors are homoskedastic, which is unrealistic because the logarithm of the observed mortality rate is much more variable at older ages.

Regardless of the method of projection used, when projecting mortality at very old ages several problems arise, in particular, because of inaccuracies in the data available and variability due to small exposures to risk. As a result, the methodologies mentioned in this chapter usually restrict the range of ages under study.

Chapter 3

Analysing the distribution of life annuities using simulation techniques

3.1 Introduction

Projecting the behaviour of future mortality rates is a complicated process given that there are many factors that are likely to affect future mortality rates and that the effect of some of these (for example, social, economic, cultural and ethnic factors) may be difficult to measure or even to model. These factors affect different people differently, which makes the process of forecasting the future course of mortality change a challenge.

In this chapter we aim to investigate the effect of mortality improvements on the expected costs of annuities using simulation techniques, which are used to facilitate the investigation of the properties of the distribution of outcomes rather than limiting it to the expected values. These are useful in their own right.

However, computer power has increased considerably, and with the rapid progress in computer technology and the decline in the real price of the hardware and the software, these have made simulation methods a cost-effective way for

representing the uncertainty associated with many actuarial problems and it is thus possible to investigate more properties of the distribution of outcomes.

In general, simulation methods offer a very powerful tool for handling actuarial problems, as they allow the modelling of various scenarios that provide a spread of results and allow the computation of the likelihood of the outcomes. So we can use simulation techniques to model a particular path that a group of persons may follow during their lifetime, by allowing the time of death of each insured in the portfolio to be a random value, and then we will be able to obtain the underlying distribution of annuity payments of the whole group. Hence we can comment on the effect of the mortality risk in a life annuity portfolio.

As a part of investigating mortality risk, factors affecting mortality risk need to be investigated as well. These include but are not limited to; age at inception, gender, assumed interest rate and level of mortality assumed.

This investigation is performed as an attempt to answer questions such as: given the improvement in future mortality rates, which age ranges will contribute the most to the expected changes in annuity values, and what is the effect of initial age, gender, rate of interest and the level of mortality improvement on the additional cost implied?

The mortality projection model of Sithole et al (2000), which is of type d). (see section 2.2), has been used in this chapter to allow for mortality improvements for both male and females pensioners. The results are illustrated using UK data from the Continuous Mortality Investigation Bureau (1998).

The chapter is organized as follows. In section 2, a description of the mortality projection model of Sithole et al (2000) is given. In section 3 a description of the simulation procedures and methodology used will be given in detail and the results of the simulations are then summarised and compared with those obtained using the analytical approach. In order to confirm the results obtained in section 3, Section 4 will deal with testing the sensitivity of the results with regard to the different factors that are likely to affect the mortality risk in a life annuity. In this study the effect of age, gender, assumed interest rate and the level of mortality improvement on the mortality risk and hence on the cost of annuity is investigated. Lastly, section 5 discusses the overall conclusions and makes some recommendations.

3.2 Mortality projection models of Sithole et al (2000)

These models are an application of the structure suggested by Renshaw et al (1996), which is itself an extension of the “Gompertz-Makeham” (GM) formula used by the Continuous Mortality Investigation Bureau (CMIB), with an age specific trend adjustment added. Using the framework proposed by Renshaw et

al (1996), the equation representing the force of mortality at age x in year t is as follows:

$$\mu_{x,t} = \exp\left(\beta_0 + \sum_{j=1}^s \beta_j L_j(x')\right) \exp\left(\sum_{i=1}^r \left(\alpha_i + \sum_{j=1}^s \gamma_{i,j} L_j(x')\right) t'^i\right) \quad (3.1)$$

subject to the convention that some of the $\gamma_{i,j}$ maybe pre-set to 0. x' , t' denote the age and time variables which have been transformed linearly and mapped on to the range $[-1,1]$. L_j is a Legendre polynomial generated by

$$L_0(x) = 1, \quad L_1(x) = x, \quad (n+1)L_{n+1}(x) = (2n+1)xL_n(x) - nL_{n-1}(x)$$

where n is an integer and $n \geq 1$.

From equation (3.1), it can be seen that the first multiplicative term takes the form of a $GM(0, s)$ formula. The second term maybe interpreted as an age specific trend adjustment term, provided that at least one of the $\gamma_{i,j}$ terms is not pre-set to zero. It is the product of r expressions that are very similar to a $GM(0, s)$, with the difference that now each exponent is multiplied by a power of t' . The optimum values of r and s can be obtained by comparing the improvement in the scaled deviance, resulting from successive increases in the values of r and s , with critical values for the χ^2 distribution with one degree of freedom. The optimum

values should be those after which the improvement in the deviance is not statistically significant. Full details are given in Sithole et al (2000).

One difficulty that can arise is ensuring that the fitted model also leads to projected mortality rates that have a good shape. Thus, the model that provides the best fit to the historic observed data is not essentially the one to be used, since the smoothness, shape and the suitability of the model to be used for projections have to be taken into consideration.

Sithole et al (2000) have developed models for projecting mortality improvements for two data sets (i.e. insured annuitants and pensioners) comprising both males and females by fitting the Renshaw et al (1996) model to CMI data. We will consider the second data set which relates to male and female life office pensioners¹ for the period from 1983 to 1996.

From the analysis of deviance and the statistical significance of additional parameters introduced in the model in equation 3.1 and after taking into consideration the general shape of the fitted curve, a 6-parameter model found to

¹ i.e. members of pension schemes administered by life insurance companies who, on retirement are compelled to annuitize: for a discussion of adverse selection in the UK market see Finkelstein and Poterba (2002).

be the best fit for both male and female life office pensioners, i.e. the model formula is

$$\mu_{x,t} = \exp\left(\beta_0 + \sum_{j=1}^3 \beta_j L_j(x') + (\alpha_1 + \gamma_{11} x')t'\right) \tag{3.2}$$

The values of the parameters can be estimated using the maximum likelihood method. Table 3.1 shows the parameter estimates for the model in 3.2 for both male and female life office pensioners.

Table 3.1: Sithole et al (2000) parameters' estimates for life office pensioners' 6-parameters models

Parameter	Parameters' Estimates	
	Female office Pensioners	Male office Pensioners
β_0	-3.1771	-2.5356
β_1	1.8380	1.8232
β_2	-0.0259	-0.2649
β_3	-0.0364	-0.0475
α_1	-0.0830	-0.1257
γ_{11}	0.0556	0.0967

After the model that provides the best fit to the data has been determined and the values of the parameters have been estimated, then projections based on the

model over a 20-year period are considered. By using both the model and the character of the resulting projections, the model can then be revised in order to produce the reduction factors that can be used subsequently.

The reduction factor model recommended here is defined in terms of a ratio of the forces of mortality rather than mortality rates as originally suggested by the CMIB. Thus, for a life attaining age x after t years from the base year, the formula for the projected force of mortality at time t will be as follows:

$$\mu_{x,t} = \mu_{x,0} RF(x,t)$$

where $\mu_{x,0}$ is the value of μ_x in the relevant 'base' table.

Using the data set mentioned above, the models for the reduction factor, for life office pensioners, that have been developed by Sithole et al (2000) are as follows:

- For Female life office pensioners:

$$RF(x,t) = \min [\exp[(-0.050651 + 0.000489x)t], 1]$$

- For Male life office pensioners:

$$RF(x,t) = \min [\exp[(-0.078846 + 0.000744x)t], 1]$$

so that when the formula-based reduction factor exceeds 1, it is set to be 1.

Table 3.2 shows the reduction factors calculated using these two models for selected ages (x) and time (t = 10) ahead of the base year.

Table 3.2: Sithole et al (2000) reduction factors

Age	Reduction Factors	
	t=10	
	Female office Pensioners	Male office Pensioners
65	0.828068	0.737227
70	0.848564	0.765168
75	0.869567	0.794168
80	0.891090	0.824268

3.3 Analysing Annuity values using simulation techniques

An Actuarial present value of an annuity represents an average of the interest discounted value of payments that is going to be received in the future, and even if the actuarial assumptions regarding interest rates have been met exactly, there will be a variation due to both Mortality and Longevity risks. Longevity risk mainly affects living benefits, especially at older ages, where a small percentage

difference in mortality at these ages translates into substantial extra cost. Hence, the calculations of expected values require an appropriate mortality projection in order to avoid under estimation of future costs. However, the projection itself is affected by uncertainty, since future mortality changes are neither known nor easily predictable. In this chapter, we attempt to quantify this risk in various circumstances using simulation techniques, so that the whole distribution of annuity payments can be considered and risk measures such as standard deviation and coefficient of variation can be used.

3.3.1 Methodology

For both males and females, the calculations are based on a single life annuity with payments of £1 due at the end of each year and an interest rate of 6% pa payable to a person aged 60 years. The cost of the annuity is calculated to be the present value of payments made to the members of the portfolio. For each policyholder and for each year we generate a random number from a uniform (0,1) distribution. If the q -type probability of the policyholder is smaller than this number, we consider that the policyholder survives and we then record that the relevant survival payment is made for that year. If the policyholder survives for that year, another random number is generated and, if again this is bigger than the q -type probability for that interval, it is assumed that the policyholder survives again and the relevant payment is recorded. This process is continued

until the policyholder dies. All the payments made to this policyholder during his/her lifetime are then recorded and the present value (at outset) of the annuity payments received, for each policyholder, is calculated. This is carried out for all n policyholders. We now have a sum comprising all of the discounted payments made to the group of policyholders. It is assumed that the policyholders are all of the same age, and hence the n simulations can be regarded as applying to one single policyholder instead of one simulation for each of n policyholders. (These two approaches should lead to the same result.)

For both male and female office pensioners, the simulation results for the two different sets of mortality rates are compared; the (1991-1994) life office pensioners tables without allowing for any future mortality improvement and the (1991-1994) life office pensioners tables with mortality improvement being allowed for using the log-link model suggested by Sithole et al (2000). In all cases, the maximum number of simulations that can be handled by Excel (65,000) has been used.

3.3.2 Simulation results

3.3.2.1 Simulated distributions based on various mortality bases for female pensioners aged 60

As mentioned above, a single annuity with annual payment of £1 payable each year to a female life office pensioner aged 60 is considered and a 6% pa interest rate is used for discounting annuity payments.

Projected rates of mortality are produced by applying the reduction factors developed by Sithole et al (2000) to the q -type mortality rates from the base table ((1991-1994) mortality table). An adjustment is needed, as the reduction factor is defined as a ratio of the forces of mortality rather than mortality rates. The approximation given by Waters and Wilkie (1987) for q_x as a function of μ_x is used.

$$q_x = \frac{0.5(\mu_x + \mu_{x+1})}{1 + 0.5\mu_{x+1}} \quad (3.3)$$

Table 3.3 shows, the summary of the descriptive statistics of the distribution of present values of annuities under the two mortality bases assumed; the (1991-1994) life office pensioners tables without allowing for any future mortality improvement (case A) and the (1991-1994) life office pensioners tables with mortality improvements being allowed for using the log-link model suggested by Sithole et al (2000) (case B). The table also shows the analytical value for the expected present value (EPV) which has been calculated in each case.

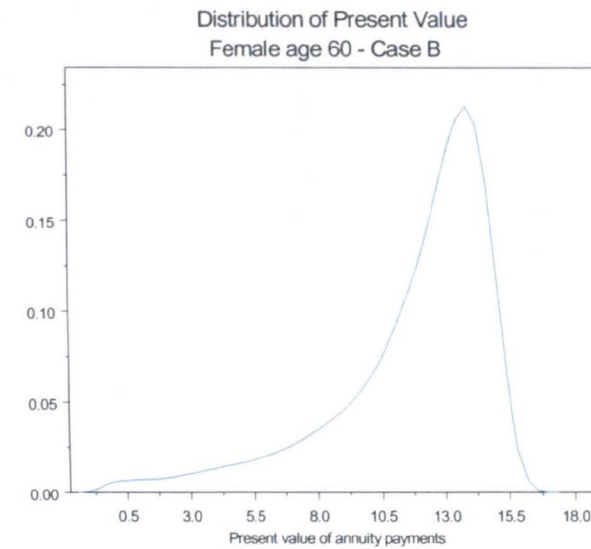
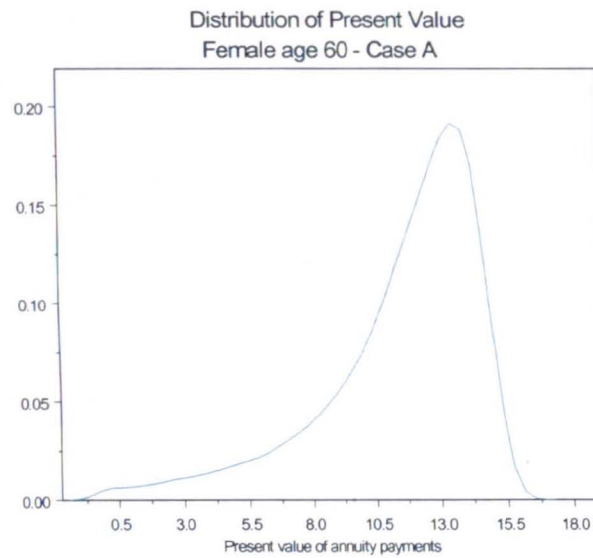
Table 3.3: Summary of the descriptive statistics of the simulated distributions- female pensioners aged 60

Age and mortality basis	(1) Mean value	(2) Standard deviation	(3) Coefficient of Variation	(4) Skewness	(5) Analytical EPV	 (1)-(5) /(5) Percentage Error
60, A	11.411	3.1090	27.24%	-1.37	11.40663	0.04%
60, B	11.762	3.1370	26.67%	-1.50	11.76808	0.05%

From table 3.3 it can be seen that the mean value of the distributions under the two different mortality bases is close to the analytical value obtained using a deterministic approach. This can be confirmed by the fact that the percentage error in the mean value is less than 0.1%. The standard deviation for the distribution of annuity values when mortality improvements are allowed for using Sithole et al reduction factors (basis B) is higher than the corresponding one under basis A. However, the coefficient of variation – which is a more meaningful and reliable measure of variability- is lower for basis B than it is for basis A, reflecting the fact that the lighter the mortality assumed the less dispersed is the distribution of annuity payments and the more reliable the mean as an estimate of the annuity payments. Also, as expected, the distribution under basis B is more skewed to the left than the one for basis A.

A graphical presentation of the simulated distributions based on the two different mortality bases is shown in figure 3.1.

Figure 3.1: Distribution of annuity payments for female pensioners aged 60 based on two mortality bases



We observe , as we have mentioned above, that the basis allowing for mortality improvements produces a distribution that is more skewed to the left than the distribution obtained by using the (1991-1994) mortality tables with no allowance for mortality improvement. This reflects a higher expected present value of the annuity when we allow for mortality improvements.

3.3.2.2 Simulated distributions based on various mortality bases for male pensioners aged 60

As for female pensioners, a single annuity with annual payment of £1 payable each year to a male life office pensioner aged 60 is considered and a 6% pa interest rate is used for discounting annuity payments. Projected rates of mortality are produced by applying the reduction factors developed by Sithole et al (2000) to the q-type mortality rates from the base table ((1991-1994) mortality table using the same approximation as before.

Similarly to table 3.3, table 3.4 shows, the summary of the descriptive statistics of the distribution of present values of annuities under the same two mortality bases used before for female pensioners; the (1991-1994) life office pensioners tables without allowing for any future mortality improvements (basis A) and the (1991-1994) life office pensioners tables with mortality improvement being allowed for using the model suggested by Sithole et al (2000) (basis B).

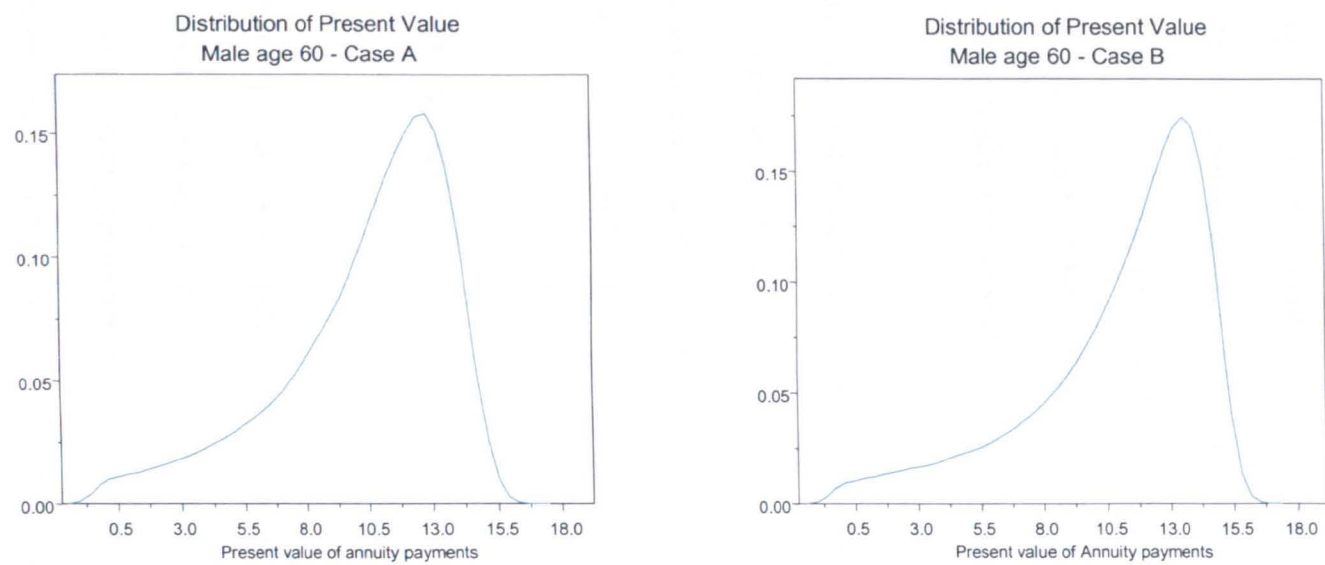
Table 3.4: Summary of the descriptive statistics of the simulated distributions- male pensioners aged 60

Age and mortality basis	(1) Mean value	(2) Standard deviation	(3) Coefficient of variation	(4) Skewness	(5) Analytical EPV	 (1)-(5) (5) Percentage Error
60, A	10.320	3.350	32.46%	-1.04	10.32564	0.05%
60, B	10.967	3.463	31.58%	-1.19	10.97443	0.07%

From table 3.4, it can be seen that, as for female pensioners, the mean value of the distribution for all ages under the two different mortality bases is close to the analytical value. Again, this can be confirmed by the fact that the percentage error in the mean value is less than 0.1%, as it is for the case of female pensioners. We observe that the coefficient of variation for male pensioners is higher under the two mortality bases than the corresponding values for female pensioners. This reflects a higher variability in the distribution of the present value of annuity payments in the case of male pensioners. Also, as expected, the mean of the distribution of annuity payments for the two bases is less than the corresponding ones for female pensioners, reflecting the higher expected present value of the annuity value for female life office pensioners. As for female life office pensioners, the coefficient of variation is lower for basis B than it is for basis A, and the distribution of annuity payments under basis B is more skewed to the left than the one for basis A.

A graphical presentation of the simulated distributions based on the two different mortality bases is shown in figure 3.2.

Figure 3.2: Distribution of annuity payments for male pensioners aged 60 based on two mortality bases



It can be seen that, the basis allowing for mortality improvements for male pensioners produces a distribution that is more left skewed than the distribution obtained by using the (1991-1994) mortality tables with no margin for mortality improvements. Again this reflects a higher present value of annuities after allowing for mortality improvements. It is also worth mentioning that the level of the skewness for the two mortality bases is less than the corresponding levels for female pensioners. This effect can be attributed to the lower mortality rates for females which lead to higher expected annuity values.

Figure 3.2 shows that the distributions are somewhat more spread out than the corresponding ones for female pensioners (Figure 3.1), confirming the conclusion obtained above regarding the higher variability in the distribution of annuity payments for male pensioners.

3.4 Sensitivity Analysis

The value of annuity payments is dependent on many factors, such as age, rate of interest and the assumed level of mortality. Hence, the effect of these factors on the distribution of annuity payments needs to be investigated. Incorporating mortality improvements is essential but not enough by itself, as allowing for mortality improvement using a certain model does not mean that annuity and pension providers are protected against longevity risk. In this section, we extend

the analysis of McCrory (1986) and undertake sensitivity tests in order to investigate how the performance of the model varies with changes in age, interest rate and the parameters of the mortality model. In each case, the distributions comparable to those in figure 3.1 and 3.2 have been derived.

3.4.1 Changes in the distribution of annuity values with age

This section investigates for each of the two bases A and B mentioned above how the distribution of annuity payments changes with age for both male and female life office pensioners. Since we are interested in analysing the mortality risk for pensioners, the ages that have been considered for sensitivity testing purposes are those above age 60. Simulated distributions of annuity payments for the two bases have been considered for ages 70 and 80 at inception.

Table 3.5 shows, for ages 70 and 80 for both male and female life office pensioners, the summary of the descriptive statistics of the distribution of present values of annuities under the same two mortality bases used before; the (1991-1994) life office pensioners tables without allowing for any future mortality improvements (basis A) and the (1991-1994) life office pensioners tables with mortality improvement being allowed for using the model suggested by Sithole et al (2000) (basis B).

Table 3.5: Summary of the descriptive statistics of the simulated distributions for different ages at inception

	Age and mortality basis	(1) Mean value	(2) Standard deviation	(3) Coefficient of Variation	(4) Skewness	(5) Analytical EPV	 (1)-(5) (5) Percentage Error
Female Office Pensioners	70, A	8.7061	3.5320	40.57%	-0.64	8.715579	0.11%
	70, B	8.9734	3.6059	40.18%	-0.70	8.968883	0.05%
	80, A	5.8562	3.3292	56.85%	-0.02	5.853390	0.05%
	80, B	5.9492	3.3909	57.00%	-0.02	5.972037	0.38%
Male Office Pensioners	70, A	7.4722	3.5423	47.41%	-0.36	7.462788	0.13%
	70, B	7.8617	3.7143	47.25%	-0.41	7.881624	0.25%
	80, A	4.7579	3.1147	65.46%	0.25	4.762788	0.10%
	80, B	4.9475	3.2327	65.34%	0.23	4.946035	0.03%

From table 3.5 it can be seen that, as age at inception increases, the relative differences between the distribution of annuity payments that allow for future mortality improvements (basis B) and the distribution of annuity payments without an allowance for any future mortality improvements decrease, reflecting the decreasing effect of mortality improvements as age increases. It can also be seen from the table that the additional cost arising from incorporating future mortality improvements for age 80 is still important and is more significant in the case of male pensioners than it is for female pensioners. The table also shows that the standard deviation for female pensioners aged 70 is higher than the corresponding one for female pensioners aged 60 for the two mortality bases. It is also clear that the coefficient of variation has increased dramatically for the two mortality bases at age 70 and 80 compared to the corresponding values for female pensioners aged 60. Similarly for male pensioners, the coefficient of variation has increased dramatically for the two mortality bases at ages 70 and 80 compared to the corresponding values for male pensioners aged 60. This reflects a higher relative level of variability in the distribution of the present value of annuity payments as age increases.

As for the case at age 60, we observe that the coefficient of variation for male pensioners for all ages is higher under the two mortality bases than the corresponding values for female pensioners. This reflects a higher variability in the distribution of the present value of annuity payments in the case of male

pensioners. A graphical presentation of the simulated distributions based on the two different mortality bases for both female and male life office pensioners is shown in figures 3.3 and 3.4 respectively.

Figure 3.3: Distribution of annuity payments for female pensioners based on two mortality bases for ages 70 and 80

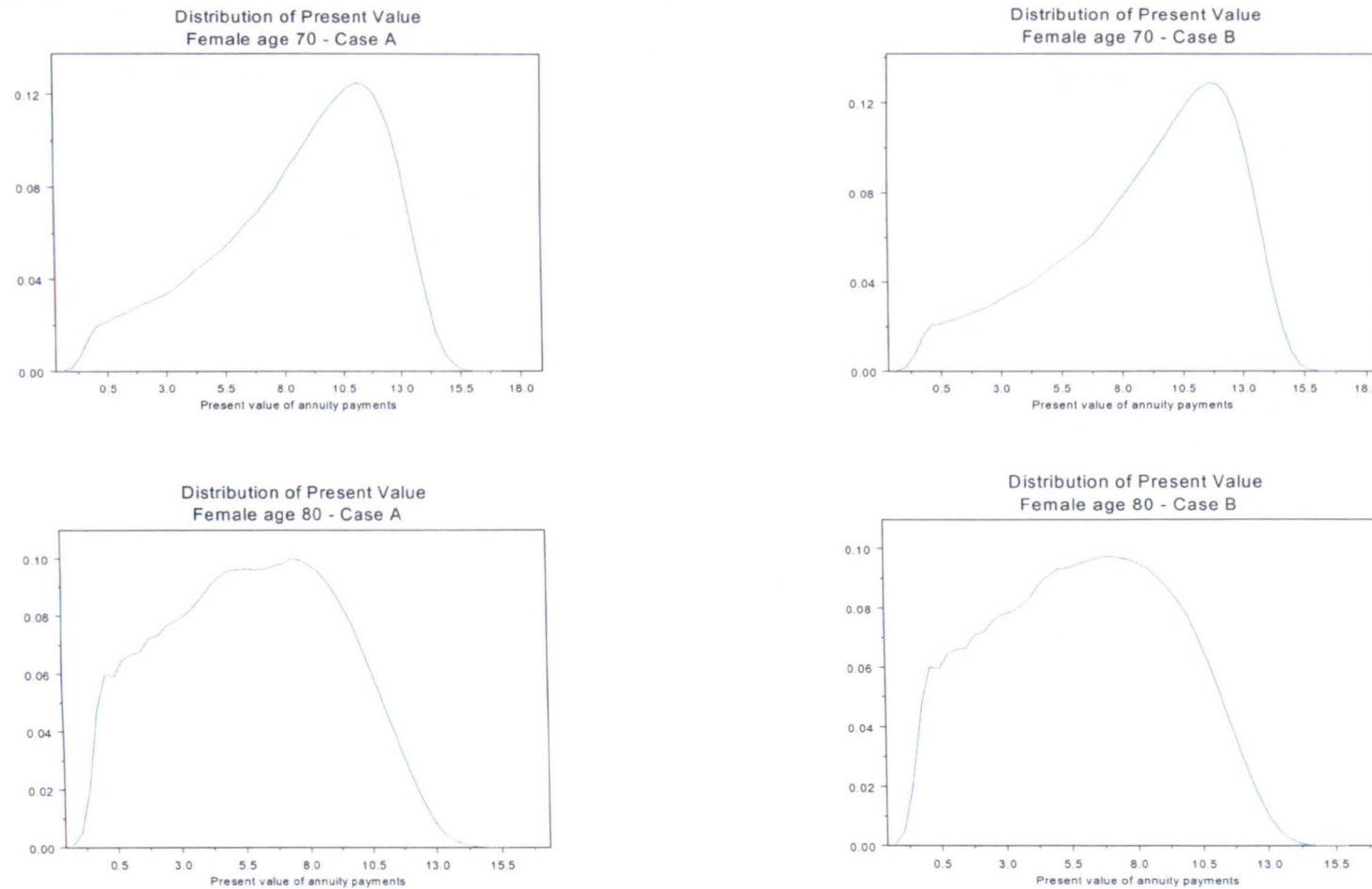
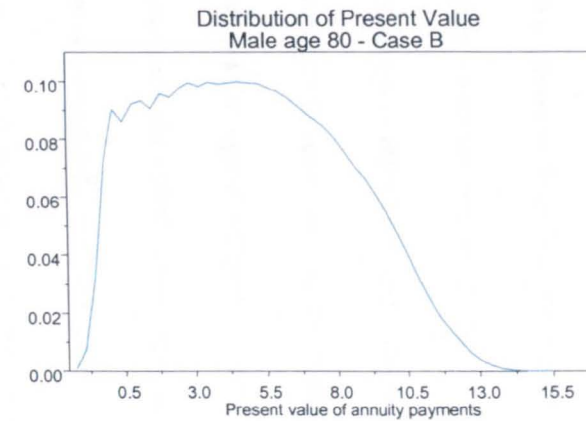
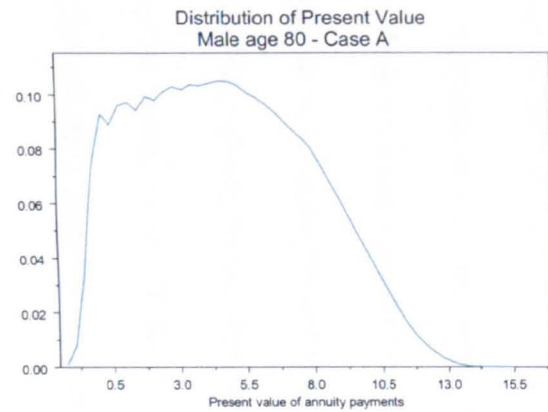
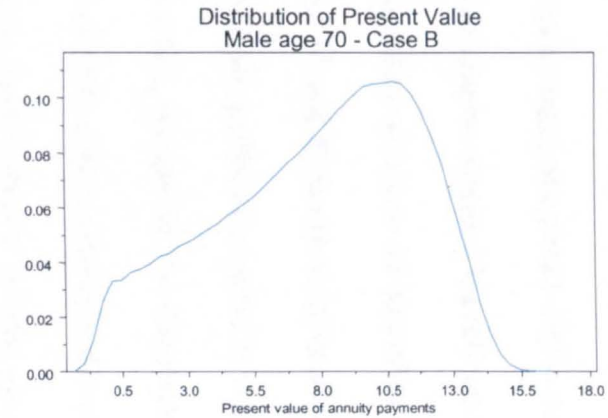
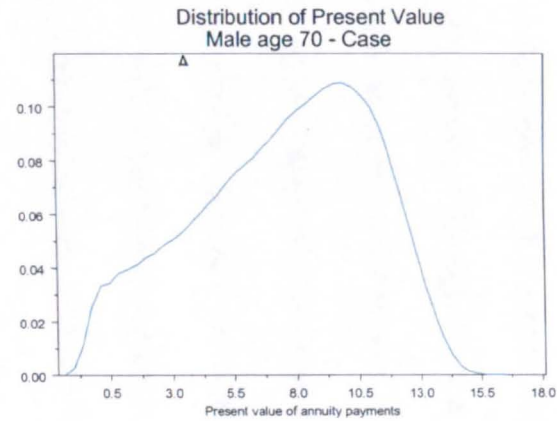


Figure 3.4: Distribution of annuity payments for male pensioners based on two mortality bases for ages 70 and 80



We observe that, for all ages, the basis allowing for mortality improvements produces a distribution that is more skewed to the left than the distribution obtained by using the (1991-1994) mortality tables with no allowance for mortality improvement. This reflects a higher expected present value of the annuity when we allow for mortality improvement. However, the difference for ages 70 and 80 is not as great as it is for age 60, an effect which reflects the decreasing effect of mortality improvements as age increases. Moreover, we note that, for age 80, the distribution based on the (1991-1994) mortality tables exhibits a very similar pattern to that obtained after allowing for mortality improvements, as after age 80 the effect of mortality improvement is of lesser significance. In general, under the two mortality bases, the level of left skewness for ages 70 and 80 is less than for age 60 as the distribution of the present value of the annuity payments shifts to the left (with the expected value decreasing) as age increases.

Also we note that for male pensioners at age 80, the two distributions are positively skewed. Although this feature is not observed for the case of female pensioners at age 80, it is not unexpected. Thus, we expect the distribution of the present value of annuity payments to change from a negatively skewed to a positively skewed distribution at some age. This reflects the fact that the present value of annuity payments decreases as age increases. In fact, at age 80, both the distribution of the present value allowing for mortality improvements and the

distribution based on the (1991-1994) mortality tables, which does not allow for mortality improvements, exhibit a similar pattern. This is because, after age 80, the effect of mortality improvements is of lesser significance.

Figures 3.3 and 3.4 show that for all ages the distributions are somewhat more spread out than the corresponding ones for female pensioners, reflecting a higher variability for male pensioners.

3.4.2 Changes in the distribution of annuity values with the rate of interest

This section investigates for each model how the difference in the value of an annuity changes with changes in the interest rate. The simulated distributions of annuity values for both males and females life office pensioners have been produced at rates of interest 2%, 4% and 8%. For both male and female life office pensioners and at all the different assumptions of interest rates a single annuity with annual payment of £1 payable each year to pensioner aged 60 is considered, Mortality rates assumed are the projected rates of mortality produced by applying the reduction factors developed by Sithole et al (2000) to the q-type mortality rates from the base table ((1991-1994) mortality table).

Table 3.6 shows, at different rates of interest, for both male and female life office pensioners aged 60, the summary of the descriptive statistics of the distribution

of present values of annuities under the mortality basis (B); the (1991-1994) life office pensioners tables with mortality improvement being allowed for using the model suggested by Sithole et al (2000).

Table 3.6: Summary of the descriptive statistics of the simulated distributions for different rates of interest

	Rate of Interest	(1) Mean value	(2) Standard deviation	(3) Coefficient of Variation	(4) Skewness	(5) Analytical EPV	(1)-(5) /(5) Percentage Error
Female Office Pensioners	2%	18.325	6.456	35.23%	-0.68	18.30006	0.136%
	4%	14.469	4.389	30.33%	-1.10	14.45414	0.103%
	6%	11.762	3.137	26.67%	-1.50	11.76808	0.052%
	8%	9.8343	2.329	23.68%	-1.91	9.827503	0.069%
Male Office Pensioners	2%	16.617	6.718	40.43%	-0.48	16.59377	0.140%
	4%	13.326	4.715	35.38%	-0.85	13.31154	0.109%
	6%	10.967	3.463	31.58%	-1.19	10.97443	0.068%
	8%	9.264	2.631	28.40%	-1.52	9.257328	0.072%

From table 3.6 it can be seen that, as the interest rate increases, the mean value of the distribution of annuity values decreases and more skewed to the left is the distribution of annuity values. As expected, this effect arises because at higher rates of interest, the effect on the expected present value of the annuity of mortality improvement is reduced by the greater discount applied to future payments (see McCrory, 1986). It can also be seen that (as is well known) the cost of life annuity is very sensitive to the rate of interest used in the calculation, with the mean value of the simulated distributions for a 2% interest rate being almost double the corresponding one for a rate of interest of 8% for female pensioners. We notice similar behaviour but with a lesser effect for male pensioners, indicating that the lower the mortality level the more sensitive the distribution of annuity payments to the changes of the rate of interest.

Table 3.6 also shows that for both male and female pensioners and for all scenarios of interest rates used, the lower the rate of interest the higher the standard deviation and more importantly the higher the coefficient of variation of the distribution of annuity payments. This indicates higher variability in the distribution of annuity payments at lower rates of interest for the same level of mortality assumed. A graphical presentation of the simulated distributions based on the different interest rate bases for both male and female life office pensioners is shown in figure 3.5.

Figure 3.5: Distributions of annuity payments for female and male pensioners aged 60 at different rates of interest

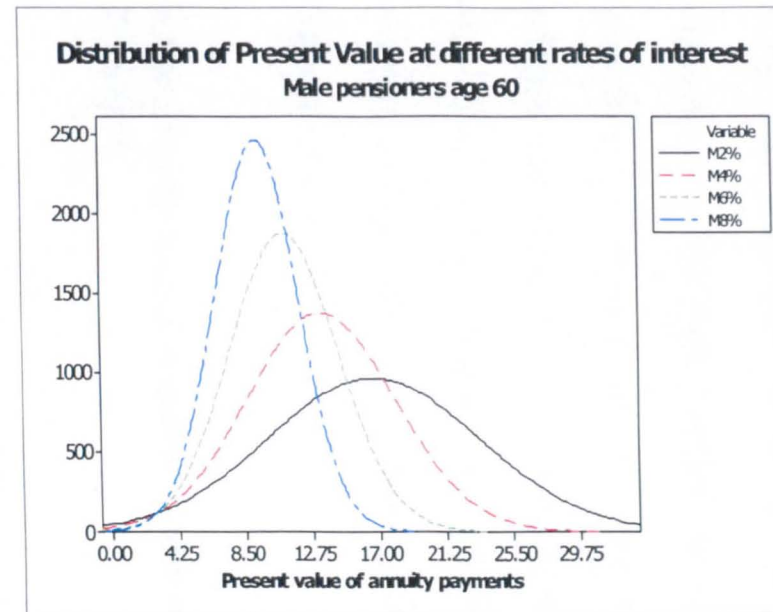
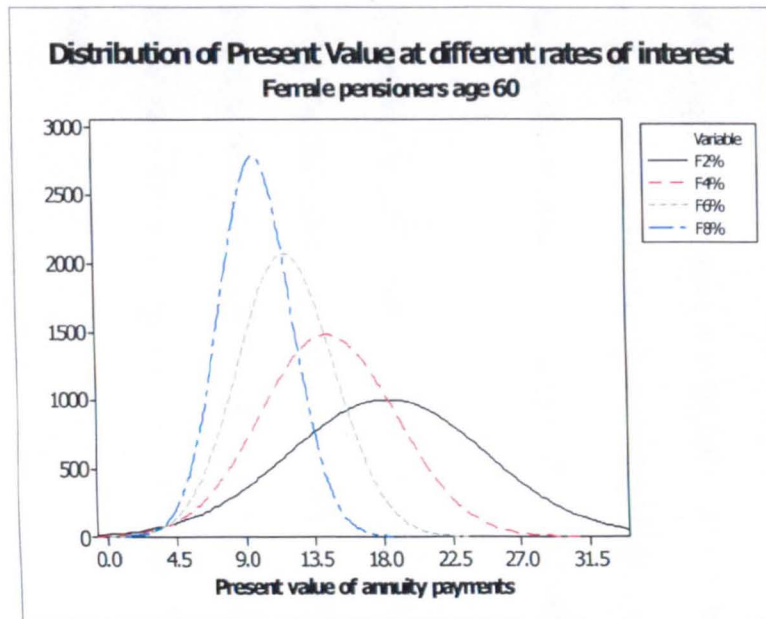


Figure 3.5 displays the distribution of present values for rates of interest 2, 4, 6 and 8% for both male and female pensioners. It is clear from the graphs that the distribution is much more spread at lower rates of interest, and as the rate of interest increases the distribution of annuity payments becomes more peaked- this confirms the conclusion regarding the higher variability of the distribution of annuity payments at lower interest rates. This means that, for a given age and level of mortality, we would have much more confidence in the calculations of the present value at higher rates of interest.

3.4.3 Changes in the distribution of annuity values with level of Mortality assumed

This section investigates the effect of changing the parameters of the mortality projection model (in a particular way) on the distribution of annuity values. We consider the effect of a change in the level of mortality improvement used in the mortality projection model. All changes are compared with the original reduction factors implied by the models of Sithole et al described in section 2.

Two cases will be considered, a 35% increase in the reduction factors, and a 35% decrease in the reduction factors (over a 65 year time horizon for an individual initially aged 55) for both males and females, Under the two cases, if we define a to be the reduction factor for a life attaining age x and after a period of t years from the base year, then

$$a = RF(x, t) = \exp[(\alpha + \beta x)t]$$

a^* is defined as

$$a^* = \exp[(\alpha^* + \beta^* x)t] = (1 + c)a$$

where c is equal to 0.35 (assuming a lower level of mortality improvement) or -0.35 (assuming a higher level of mortality improvement) when $x=55$ and duration $t=65$. The value of α^* can be found as follows:

$$\alpha^* = \log\left(\frac{a^*}{t}\right) - x\beta^* \quad (3.4)$$

If we assume also that $\beta = \beta^*$, then (3.4) can be expressed as:

$$\alpha^* = \log\left(\frac{(1+c)a}{65}\right) - 55\beta \quad (3.5)$$

where $a = RF(55, 65)$.

The revised reduction factor, $RF^*(x, t)$, for each model will be as follows:

- For Female life office pensioners:

$$c = 0.35 \quad RF^*(x, t) = \exp[(-0.046034006 + 0.000489x)t]$$

$$c = -0.35 \quad RF^*(x, t) = \exp[(-0.057278429 + 0.000489x)t]$$

- For Male life office pensioners:

$$c = 0.35 \quad RF^*(x, t) = \exp[(-0.074229006 + 0.000744x)t]$$

$$c = -0.35 \quad RF^*(x, t) = \exp[(-0.085473429 + 0.000744x)t]$$

As before, we consider a life aged 60 and interest rate of 6% under each model.

Table 3.7 shows, for both male and female life office pensioners aged 60, the summary of the descriptive statistics of the distribution of present values of annuities under the (1991-1994) life office pensioners tables with mortality improvement being allowed for using the two mortality improvements bases; a 35% increase in the reduction factors of the model suggested by Sithole et al (2000) (case 1), and a 35% decrease in the reduction factors of the model suggested by Sithole et al (2000) (case 2).

Table 3.7: Summary of the descriptive statistics of the simulated distributions for different scenarios of mortality improvements

	Mortality basis	(1) Mean value	(2) Standard deviation	(3) Coefficient of variation	(4) Skewness	(5) Analytical EPV	 (1)-(5) /(5) Percentage Error
Female Office Pensioners	1	11.636	3.101	26.65%	-1.49	11.626	0.086%
	2	11.989	3.173	26.46%	-1.54	11.981	0.067%
Male Office Pensioners	1	10.838	3.412	31.48%	-1.18	10.829	0.083%
	2	11.202	3.517	31.40%	-1.22	11.190	0.107%

Table 3.7 shows, as expected, that the mean of the distribution of annuity payments is higher at lower level of mortality assumed (case 2) for both male and female pensioners. The distribution of annuity payments is more skewed to the left under (case 2) where greater improvement in mortality is assumed than the corresponding one under (case1) for both male and female pensioners. Again the coefficient of variation is lower for mortality improvement basis (case 2) for both genders, which verifies the conclusion we had earlier that the coefficient of variation is lower for the lighter mortality group which reflects lower variability in the distribution of annuity payments for this group.

A graphical presentation of the simulated distributions based on the two different bases for mortality improvement for both male and female life office pensioners is shown in figure 3.6.

Figure 3.6: Distributions of annuity payments for female and male pensioners aged 60 for different scenarios of mortality improvements

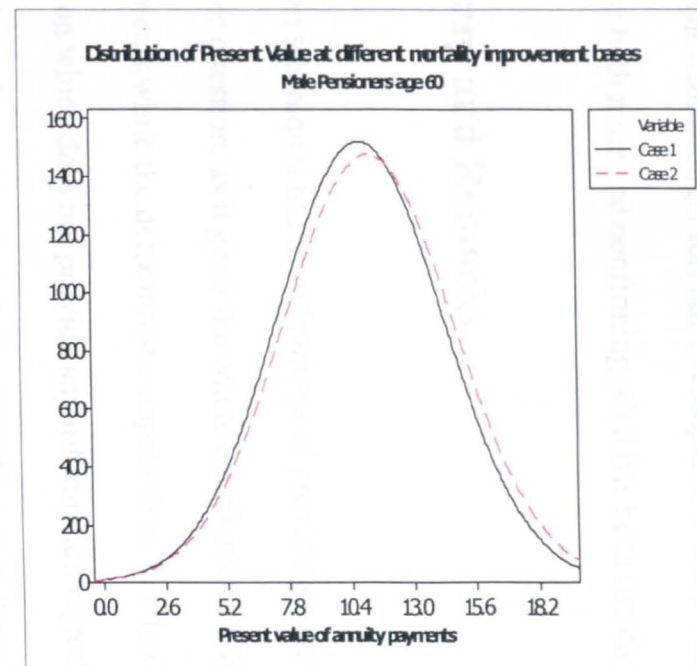
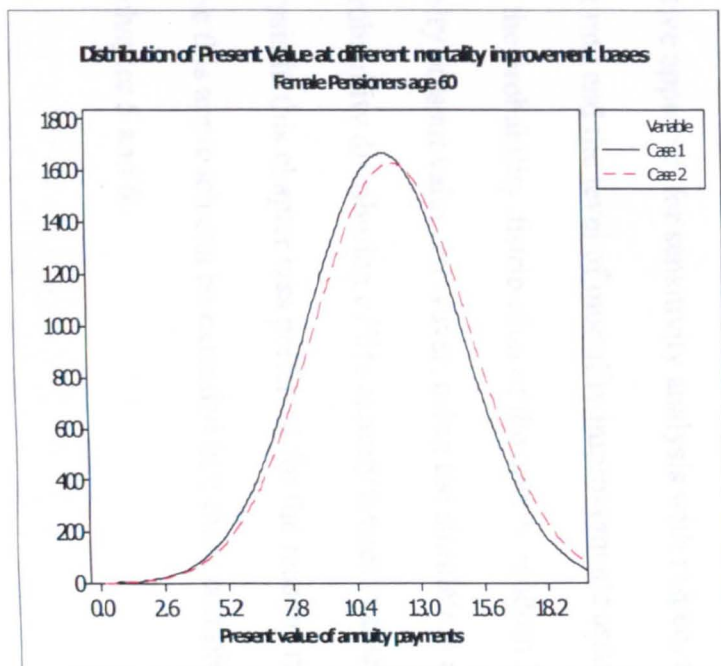


Figure 3.6 shows that the distribution of annuity payments under basis 2 is shifted to the right with a longer left tail as compared to the corresponding one under basis 1 for both male and confirming what has been mentioned earlier.

3.5 Summary and Remarks

Using simulation techniques has the advantage of providing a better assessment for the risk under question, as it gives the whole distribution of the present value of annuity payments while the deterministic approach uses only the mean value of the distribution which does not provide any information regarding the level of the dispersion about the mean value. It also provides a very powerful tool in the sense that it allows modelling of different scenarios at the same time which gives a better understanding of the results obtained. However, it is worth mentioning that, an alternative approach for sensitivity analysis with respect to age at entry, gender, interest rate and the level of mortality improvements assumed would be by considering the probability distribution of the curtate random life time of individual annuity present value. However, using the simulation approach to construct the probability distribution of life annuity present value and to perform sensitivity analysis in this chapter was preferred for the reasons mentioned above and also because this approach can be extended into more complex contexts as it can be seen in chapter 5 and 6.

Using the simulated distributions, we can draw some conclusions regarding the mortality risk in the context of a life annuity portfolio:

- Generally, there is less variation in the distribution of the present value of annuity payments for female pensioners than in the corresponding one for male pensioners, and for both genders the coefficient of variation increases with age.
- As age at inception increases, the effect of mortality improvement decreases and the shapes of the distributions become similar to each other.
- For younger ages at inception, the shape of the distribution of present value of annuity payments is negatively skewed. With increasing age, the distribution becomes less negatively skewed until, at some point, it becomes positively skewed, reflecting the fact that the expected present value of annuity payments is lower for older ages. This effect has been shown for males at around age 80.

As we have mentioned in section 3.4 the distribution of annuity payments is very sensitive to the rate of interest used in valuations. This means that the lower is the rate of interest, the higher is the effect of the longevity risk on the present value of annuity payments. In other words, the effect on the present value of annuity payments of living longer than average is increased by the lower discounting applied to the future payments in a low interest environment.

As we have also mention in section 3.4 the distribution of annuity payments is very sensitive to the level of mortality improvement assumed valuations. It has been also noted that decreasing the reduction factors (i.e. assuming higher mortality improvements.) (Case 2) has a stronger effect on the additional cost of annuity than increasing the reduction factors by the same percentage since the mean value of distribution of annuity payments when reduction factors are as Sithole (2000) is closer to the mean value of the distribution that assumes a lower improvement in mortality (case 1) than the one assumes a higher improvement in mortality (case 2).

Chapter 4

Entropy, Longevity and the Cost of Life Annuity

4.1 Introduction

The classical concept of entropy has been found to be very useful for describing the information available in living systems in a variety of contexts and is considered to be the bedrock of classical information theory (introduced by Shannon, 1948). For a concise discussion of the important mathematical concepts of information theory and entropy, see Khinchin (1957). One application of information theory is the concept of population entropy as a measure of the diversity of the population introduced by Demetrius (1976).

Many demographers have considered analyzing the change in life expectancy over time. Pollard (1982, 1988), Arriaga (1984), Pressat (1985) and Andreev (1982, 2002) focused on discrete difference in life expectancy at two moments in time, while Keyfitz (1977) considered continuous changes in life expectancy and derived a formula that relates the time-derivative of life expectancy to the entropy of life table survivorship where the entropy measure is used as an index

to measure the effect of proportional change in the force of mortality on life expectancy. This approach extends the definition of the entropy of a population that has been derived and applied in population biology by Demetrius (1976) to measure the variability of the contribution of the different age classes to the stationary age distribution.

In this chapter, the formula derived by Keyfitz (1977) will be extended to measure the effect of any changes in the force of mortality on the cost of life annuity for different scenarios of interest rates and levels of mortality improvements. This will allow different sources of risk in a life annuity contract to be summarized in a single figure index.

The chapter is organized as follows. In section 4.2, a description of the definition of entropy in demography and how it can be extended to measure the effect of proportional change in the force of mortality on the cost of life annuity is given. In section 4.3, numerical values for the entropy measure for life annuities over the whole age range are obtained and analysed for different interest rates using English life tables over the period from 1851 to 1991 for both males and females. In section 4.4, numerical values for the entropy measure for life annuities are derived for different interest rates using different mathematical models for mortality projection for both male and female aged 60 as an attempt to gain a better understanding regarding the properties of the entropy measure. The

mortality models that are used in section 4.4 are: the Gompertz model and the Sithole et al (2000) model that has been described earlier in chapter 2. Section 4.5 will deal with testing the sensitivity of the results obtained in section 4.4 with regard to the different factors that are likely to affect the value of the entropy measure. In this study, the effect of gender, assumed interest rate and the level of mortality improvement on the mortality are investigated. Lastly, section 4.6 discusses the overall conclusions and remarks.

4.2 Entropy and Mortality

In this section, we will derive the formula for the entropy measure (H) for both the life expectancy at birth as obtained by Keyfitz (1977), and the corresponding formula in the case of a life annuity.

4.2.1 Entropy for life expectancy at birth

Entropy as defined by Demetrius (1976) is a single figure index used to measure the effect on life expectancy at birth due to a proportional change in the force of mortality over the whole age range.

Suppose that the force of mortality μ_x at age x is multiplied by $(1 + \varphi)$, so that

$\mu_x^* = \mu_x \times (1 + \varphi)$, where φ is a constant change in the force of mortality at all

ages Then the new probability of surviving till age x becomes

$${}_xP_0^{\bullet} = \exp\left[-\int_0^x \mu_a^{\bullet} da\right] = \exp\left[-\int_0^x (1+\varphi)\mu_a da\right] = \exp\left[-\int_0^x \mu_a da\right]^{(1+\varphi)} = {}_xP_0^{(1+\varphi)} \quad (4.1)$$

where

$${}_xP_0 = \exp\left[-\int_0^x \mu_a da\right]$$

And the new life expectancy is

$$e_0^{\bullet} = \int_0^{\omega} {}_aP_0^{(1+\varphi)} da \quad (4.2)$$

In order to find the effect of small changes in φ on the expectation of life, we consider the derivative of equation (4.2) with respect to φ

$$\begin{aligned} \frac{d e_0^{\bullet}}{d\varphi} &= \frac{d}{d\varphi} \int_0^{\omega} {}_aP_0^{(1+\varphi)} da = \frac{d}{d\varphi} \int_0^{\omega} \exp[\ln {}_aP_0^{(1+\varphi)}] da \\ &= \frac{d}{d\varphi} \int_0^{\omega} \exp[(1+\varphi) \ln {}_aP_0] da \\ &= \int_0^{\omega} \frac{d}{d\varphi} (\exp[\ln {}_aP_0 + \varphi \ln {}_aP_0]) da \\ &= \int_0^{\omega} (\ln {}_aP_0) \exp[\ln {}_aP_0 + \varphi \ln {}_aP_0] da \\ &= \int_0^{\omega} (\ln {}_aP_0) \exp[(1+\varphi) \ln {}_aP_0] da \\ &= \int_0^{\omega} (\ln {}_aP_0) {}_aP_0^{(1+\varphi)} da \end{aligned} \quad (4.3)$$

The quantity (4.3) cannot be positive, since ${}_aP_0$ cannot be greater than 1 and ${}_aP_0^{(1+\varphi)}$ is always positive.

In the neighbourhood of $\varphi = 0$ we have

$$\frac{\frac{\Delta e_0}{e_0}}{\frac{\Delta e_0}{e_0}} \cong \frac{\int_0^{\omega} [\ln {}_aP_0] {}_aP_0 da}{\int_0^{\omega} {}_aP_0 da} \varphi = -H\varphi \quad (4.4)$$

where H is the entropy measure (or information in other contexts) and can be thought of as minus the weighted average value of $\ln {}_aP_0$, weighted by ${}_aP_0$. The ratio of the integrals in (4.4) is always negative, so that H is a positive quantity. The value of H can be as low as zero if all mortality was concentrated at a certain age. Thus, as mortality improves, we would expect a larger fraction of deaths to occur at older ages, resulting in a drop in the value of H so that it becomes closer to 0. Keyfitz (1977) and Demetrius (1979) suggested that, if the force of mortality is the same at all ages, H will take a value of 1 and that will represent the maximum entropy. In other words, the value of H ranges between 0 and 1. However, as noted by other demographic researchers it is clearly possible that the entropy value can exceed unity. This could happen if the survival curve is characterized by extremely high death rates at young ages. In this case, a

reduction of mortality by a fixed factor at all ages can result in even larger gains in life expectancy, as individuals who survive the first few years of life can expect to live many more years. (see Goldman and Lord (1986, 1987) and Hakkert (1987).

Table 4.1 shows the value of the complete expectation of life at birth \dot{e}_0 and the corresponding entropy measure H for males and females in the United States over the period 1919-21 to 1959-61.

Table 4.1: Values for the complete life expectancy and the entropy measure for males and females, United States

	Males		Females	
	\dot{e}_0	H	\dot{e}_0	H
1919-21	54.49	0.3804	56.41	0.3547
1924-26	56.34	0.3401	59.01	0.3113
1929-31	57.27	0.3272	60.67	0.2942
1934-36	58.53	0.3105	62.58	0.2725
1939-41	61.14	0.2747	65.58	0.2361
1944-46	62.26	0.2632	68.11	0.2087
1949-51	65.28	0.2260	70.86	0.1823
1954-56	66.45	0.2134	72.61	0.1660
1959-61	66.84	0.2083	73.4	0.1594

Source: Computed from data in Keyfitz and Flieger (1968)

From table 4.1, we can see that the higher the expectation of life the lower the value for the entropy measure H . This is because as life expectancy at birth rises,

it becomes less sensitive to changes in the force of mortality. This is why we expect the entropy measure H to decrease over time due to mortality improvements. It can be noted from the table that the value for H is lower for females than the corresponding one for males for the same reason.

The theory developed above seeks to find the effect on expectation of life of a uniform proportional excess in the force of mortality, μ_x . This theory can be extended to find the corresponding entropy value for annuities as shown below.

4.2.2 Entropy for Annuities

Define μ_x^* to be the new force of mortality (as defined in 3.2.1), then the new value of a life annuity at age x becomes

$$\begin{aligned}
 \bar{a}_x^* &= \int_0^{\infty} {}_tP_x^* \exp[-\delta t] dt \\
 &= \int_0^{\infty} \exp\left[-\int_0^x \mu_a^* da\right] \exp[-\delta t] dt \\
 &= \int_0^{\infty} ({}_tP_x)^{(1+\varphi)} \exp[-\delta t] dt
 \end{aligned} \tag{4.5}$$

where δ is the force of interest and $\mu_x^* = \mu_x \times (1 + \varphi)$.

The effect of small change in φ on the corresponding value of life annuity at age x can be expressed in the same manner as in section 4.2.1 to be

$$\frac{d}{d\varphi} \bar{a}_x^* = \int_0^{\infty} (\log_e P_x) (P_x)^{(1+\varphi)} \exp[-\delta t] dt \quad (4.6)$$

We can then find an expression of the ratio between the value of life annuity at age x when the force of mortality is μ_x^* and the corresponding one when the force of mortality is μ_x as follows

$$f(\varphi) = \frac{\bar{a}_x^*(\varphi)}{\bar{a}_x(0)} = \frac{\int_0^{\infty} P_x^* \exp[-\delta t] dt}{\int_0^{\infty} P_x \exp[-\delta t] dt} \quad (4.7)$$

where the ratio is expressed as a function of φ , the proportional change in the force of mortality.

Equation (4.7) can then be expanded using a Taylor expansion as follows

$$f(\varphi) = f(0) + \varphi f'(0) + \frac{1}{2} \varphi^2 f''(0) + \dots \quad (4.8)$$

Using the result obtained in equation (4.6)

$$f(\varphi) - 1 = \frac{\Delta \bar{a}_x}{\bar{a}_x} = -H\varphi$$

where

$$H = \frac{- \int_0^{\infty} (\log_e P_x) (P_x) \exp[-\delta t] dt}{\int_0^{\infty} P_x \exp[-\delta t] dt} \quad (4.9)$$

Here H is the entropy measure for an annuity and can be thought of as minus the weighted average value of $\ln P_x$, weighted by $P_x \exp[-\delta t]$. The ratio of the integrals in (4.9) is again always negative, so that H is a positive quantity. As before, for a constant force of interest δ , the value of H can be as low as zero if all mortality were concentrated at a certain age. Thus, as mortality improves, we would expect a larger fraction of deaths to occur at older ages, resulting in a drop in the value of H so that it becomes closer to 0. Again as discussed earlier as mortality increases the value of H increases, but the pace by which this happens is dependant on the concavity of the survivorship curve of the population in question. Also, as the value of δ increases H is expected to decrease, as any change of the force of mortality is expected to have a lower effect on the cost of life annuity at high interest rates.

The theory developed in this section seeks to find the effect on the value of an annuity of a uniform proportional increase or decrease in the force of mortality over the age range. However, this is unlikely to be the case in practice as mortality changes in different proportions at different ages. One way to deal with this problem is to calculate the proportional change in the force of mortality that will have the same effect on the value of a life annuity as a set of different changes at different ages suggested by a mortality model.

4.3 Numerical results for H using English Life Tables

In this section the values of the entropy measure H will be calculated using English Life mortality tables over the period from 1851 to 1991, so that the improvement in mortality can be taken into account. Calculations have been done for both males and females at different rates of interest. The rates of interest that have been used are 0%, 2%, 4%, 6% and 8%. Three scenarios for the age range have been considered in the calculations namely, calculating H over the whole age range (i.e. 0 – 110) and a subset of the age range (60-110 and 70-110).

4.3.1 Numerical results for H over the whole age range

In order to calculate values for H we need to calculate the ratio of integrals in equation (4.9) which is not convenient for numerical purposes, and hence an approximation is needed to make the calculations more convenient.

Define:

$${}_tQ_x = \int_0^t \mu_{x+u} du \quad \text{and} \quad {}_tE_x = {}_tP_x \exp(-\delta t) \quad (4.10)$$

In this case and for numerical evaluation purpose, we calculate ${}_tQ_x$ from:

$${}_tQ_x = -\ln\left(\frac{l_{x+t}}{l_x}\right) = -\ln {}_tP_x \quad (4.11)$$

Then, the mean value theorem for integrals can be used, and we can replace the integrals in (4.9) by a sum of one-year integrals, leading to the following approximation:

$$H \approx \frac{\sum_{t=0}^{\omega} {}_{t+\frac{1}{2}}Q_x {}_{t+\frac{1}{2}}E_x}{\sum_{t=0}^{\omega} {}_{t+\frac{1}{2}}E_x} \quad (4.12)$$

Equation (4.12) can then be used to calculate the values of H over the whole age range, i.e. when $x=0$.

Table 4.2 shows the values of the entropy measure, H , over the whole age range for females using different interest rates and English life tables over the period (1851-1991).

Table 4.2: Entropy values over the whole age range – Females

English Life Table (Females)	Interest Rate				
	0%	2%	4%	6%	8%
1991	0.13763	0.06536	0.02948	0.01630	0.01124
1971	0.15825	0.08513	0.05013	0.03527	0.02906
1951	0.18431	0.10951	0.07200	0.05497	0.04722
1931	0.29546	0.20531	0.15584	0.13011	0.11620
1911	0.41575	0.31363	0.25583	0.22418	0.20592
1891	0.54386	0.42841	0.35996	0.32021	0.29591
1871	0.60370	0.47632	0.39873	0.35206	0.32248
1851	0.64237	0.50796	0.42498	0.37429	0.34171

Table 4.2 shows that, as mortality has become lighter, the value for H has decreased, reflecting the fact that in a low mortality environment any change in the force of mortality will have a smaller effect on the cost of life annuity as compared to the corresponding effect in a high mortality environment. And, as expected, the effect is of lesser importance at higher rates of interest, as we can see that, for the same mortality table used, the value of H decreases when the rate of interest increases. Table 4.3 shows the corresponding values for H for males.

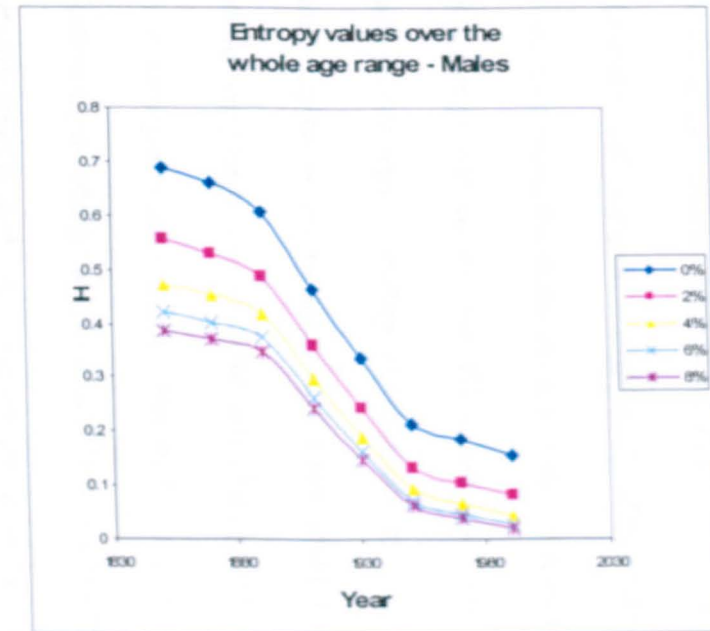
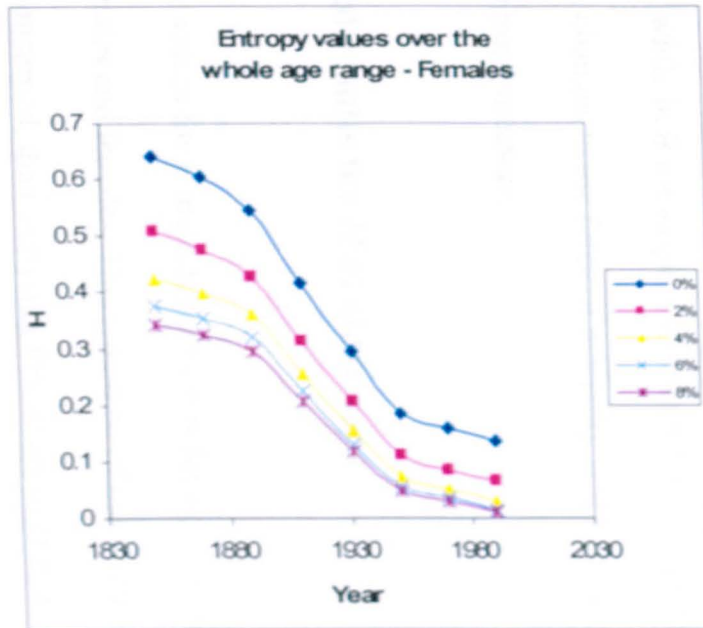
Table 4.3: Entropy values over the whole age range – Males

English Life Table (Males)	Interest Rate				
	0%	2%	4%	6%	8%
1991	0.15576	0.07926	0.04213	0.02602	0.01916
1971	0.18524	0.10556	0.06552	0.04725	0.03900
1951	0.21369	0.13309	0.09115	0.07111	0.06152
1931	0.33543	0.24201	0.18957	0.16149	0.14593
1911	0.46447	0.35892	0.29805	0.26396	0.24392
1891	0.60601	0.48756	0.41643	0.37442	0.34832
1871	0.66110	0.53117	0.45144	0.40293	0.37184
1851	0.68864	0.55498	0.47204	0.42093	0.38774

Table 4.3 shows that a similar picture emerges for males. The value of H decreases as mortality improves, and again as the rate of interest increases the value of H decreases. It can also be seen from tables 4.2 and 4.3 that at all levels of interest rates and for all mortality levels the value of H is lower for males than females.

A graphical presentation of the values for H for different English life tables and different rates of interest for both males and females is shown in figure 4.1.

Figure 4.1: Entropy values for different English life tables at different rates of interest for both females and males at age 0



In addition to the conclusions obtained earlier from tables 4.2 and 4.3, figure 4.1 shows that for both males and females, at high levels of mortality (earlier years), any change in the force of mortality does not affect the entropy measure as much (the slope is not that steep). Moreover, this effect decreases with the increase in the interest rate such that at high mortality level the curve looks almost horizontal. The same applies when mortality is at a low level. However, when the mortality is neither very high nor very low, any change in the force of mortality will have a stronger impact on the value of the entropy measure. We can express this feature by dividing the curve into three sections - one on the far left where the mortality level is high (A) , one on the far right where the mortality level is low (C) and the section in the middle where mortality is neither high or low (B). For both A and C the slope of the curve shows that the value of H is inelastic, hence, changes in the force of mortality will not have a great effect on the value of H , while in B the slope is steeper reflecting the fact that the value of H is elastic, i.e. changes in the force of mortality will have a strong effect on the value of the entropy measure.

4.3.2 Numerical results for H at older ages

In this section, the values for H are calculated at older ages, namely at ages 60 and 70, for both males and females. This would be of more interest, as far as longevity risk is concerned, than analysing the values of H that have been

calculated over the whole age range (i.e. at age 0) as it would give an idea about the effect of any change of the force of mortality on the cost of life annuity at age 60 and 70 at different rates of interest. This would be more meaningful than the corresponding effect on the cost of life annuity at age 0, which was helpful in providing a better understanding regarding the properties of the entropy measure H .

Equation (4.12) can then be used to calculate the values of H at older ages in the same manner as before but by using $x=60$ or $x=70$.

4.3.2.1 Numerical values for H for females and males – age 60

The entropy measure has been calculated for both females and males at age 60, at different rates of interest and using mortality rates from the English Life table over the same period as before.

Tables 4.4 and 4.5 show the values of the entropy measure H for females and males at age 60 using different interest rates and English Life Tables over the period (1851-1991).

Table 4.4: Entropy values for Females - age 60

English Life Table Females	Interest Rate				
	0%	2%	4%	6%	8%
1991	0.37663	0.30309	0.24542	0.20056	0.16575
1971	0.39191	0.32088	0.26432	0.21954	0.18415
1951	0.41319	0.34367	0.28772	0.24280	0.20675
1931	0.45649	0.38500	0.32696	0.27992	0.24176
1911	0.50177	0.42583	0.36423	0.31425	0.27361
1891	0.53708	0.46164	0.39980	0.34906	0.30732
1871	0.54043	0.46365	0.40088	0.34951	0.30735
1851	0.54042	0.46310	0.39999	0.34843	0.30618

Table 4.4 shows that the values of H are higher than the corresponding ones for females at age 0 reflecting the higher effect of the change in the force of mortality on the cost of life annuity in the higher mortality group. Again the higher the interest rate the lower is the effect of changes in the force of mortality on the cost of life annuity, and hence the lower are the values of H . We expect the value of H to decrease at lower levels of mortality so that for more recent life tables the values of H should be lower at all rates of interest. This is true except for the values of H that have been calculated using English life table at year 1871, where the values of H at all interest rates were slightly higher than the corresponding ones calculated using English life table at year 1851.

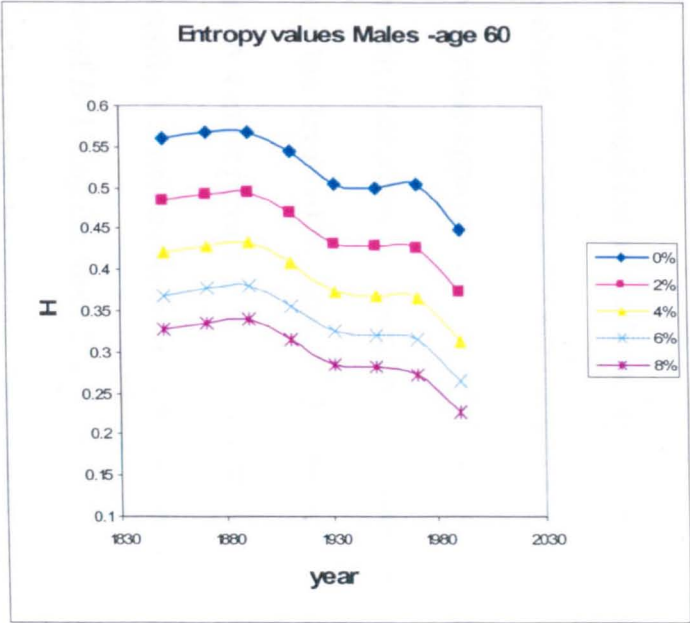
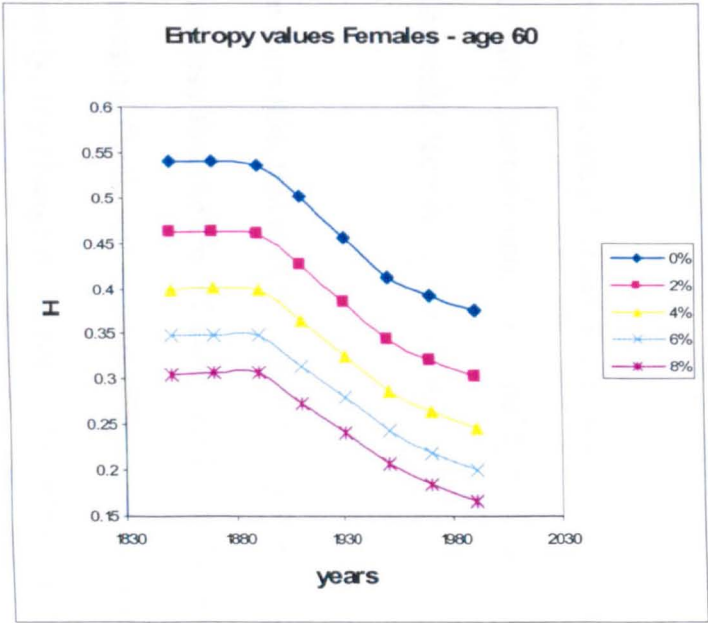
Table 4.5: Entropy values for Males - age 60

English Life Table (Males)	Interest Rate				
	0%	2%	4%	6%	8%
1991	0.44971	0.37382	0.31298	0.26434	0.22543
1971	0.50313	0.42641	0.36414	0.31361	0.27250
1951	0.49838	0.42714	0.36854	0.32038	0.28074
1931	0.50284	0.43135	0.37259	0.32428	0.28448
1911	0.54445	0.46890	0.40696	0.35610	0.31420
1891	0.56787	0.49358	0.43209	0.38113	0.33877
1871	0.56717	0.49114	0.42851	0.37686	0.33412
1851	0.55978	0.48323	0.42038	0.36871	0.32608

Table 4.5 shows that the values of H are higher than the corresponding ones for both males at age 0 and females at age 60 reflecting higher effect of the change in the force of mortality on the cost of life annuity in the group which experienced higher mortality. As before, the higher the interest rate the lower the effect of changes in the force of mortality on the cost of life annuity, hence, lower values of H . We expect the value of H to decrease at lower levels of mortality so that for more recent life tables the values of H should be lower at all rates of interest, but again this was not always the case. The values of H have increased from 1851 to 1871 and then increased again from 1871 to 1891 for all rates of interest before they have started to decrease with the improvements in mortality until 1991 - except at a rate of interest of 0% - where H has increased again in 1971.

A graphical presentation of the values for H at age 60 for different English life tables and different rates of interest for both males and females is shown below in figure 4.2.

Figure 4.2: Entropy values for different English life tables at different rates of interest for both females and males at age 60



From figure 4.2, we can see that as the rate of interest increases the value of H decreases for both males and females at all levels of mortality. Unlike the case when H was calculated over the whole age range (i.e. at age 0), the value of H does not always decrease approaching zero when mortality improves. In figure 4.1 we have seen that H decreases as mortality improves but very slowly at both very high and very low levels of mortality to the extent that the curve looks almost horizontal. A similar picture is observed here, for females at age 60 at a high level of mortality a considerable reduction in mortality is needed to decrease the value of H . Using the mortality rates of 1871 instead of 1851, although lighter, is not enough to decrease the value of H so it increased. When the reduction in mortality at this age becomes more considerable, the value of H started to decrease as mortality improves.

The same happens in the case of males at age 60 at a both very high and very low levels of mortality, improvements in mortality rates are not sufficient to be translated into a material decrease in the value of H .

We can conclude from this, that at very high and very low levels of mortality, the low value of H suggests that the cost of life annuity is not responsive to changes in the force of mortality. On the other hand when the level of mortality does not lie in either extremity, any change in the force of mortality does affect the cost of a life annuity, and this leads to a higher value for H . Hence, the value of H is

expected to be low at a very high mortality level till mortality improves enough such that the cost of life annuity is responsive to changes in mortality rates which will translate into an increase in the value of H , then any more improvements will have a lesser effect leading to a lower value for H .

4.3.2.2 Numerical values for H for females and males – age 70

The entropy measure H at age 70 has been calculated for both females and males at different rates of interest and using mortality rates from the English Life table over the same period as before.

Tables 4.6 and 4.7 show the values of the entropy measure H for females and males at age 70 using different interest rates and English Life Tables over the period (1851-1991).

Table 4.6: Entropy values for Females - age 70

English Life Table	Interest Rate				
	0%	2%	4%	6%	8%
1991	0.47921	0.41201	0.35639	0.31040	0.27233
1971	0.51125	0.44676	0.39261	0.34711	0.30882
1951	0.54388	0.48098	0.42779	0.38271	0.34438
1931	0.58315	0.51998	0.46621	0.42034	0.38109
1911	0.62284	0.55635	0.49991	0.45186	0.41079
1891	0.64625	0.58218	0.52734	0.48024	0.43965
1871	0.64680	0.58152	0.52576	0.47801	0.43694
1851	0.64823	0.58211	0.52576	0.47757	0.43621

Table 4.6 shows that the value of H decreases with the increase in the interest rates. As for the case for females at age 60, at very high level of mortality the

value of H has increased as mortality improves. This happens for years 1871 and 1891. However, we note that it takes longer at age 70 for H to start to decrease when mortality improves as compared to females at age 60. This is because at age 70 mortality rates are higher than the corresponding ones at age 60. The values of H are higher than the corresponding ones for females at age 60 and age 0 at all interest rates and for all levels of mortality.

Table 4.7: Entropy values for Males - age 70

English Life Table	Interest Rate				
	0%	2%	4%	6%	8%
1991	0.57321	0.50578	0.44899	0.40108	0.36054
1971	0.63256	0.56592	0.50928	0.46098	0.41965
1951	0.61293	0.55242	0.50043	0.45562	0.41690
1931	0.63241	0.57094	0.51812	0.47259	0.43320
1911	0.65602	0.59145	0.53625	0.48888	0.44807
1891	0.66521	0.60383	0.55085	0.50499	0.46515
1871	0.66629	0.60274	0.54815	0.50111	0.46042
1851	0.66581	0.60104	0.54558	0.49793	0.45682

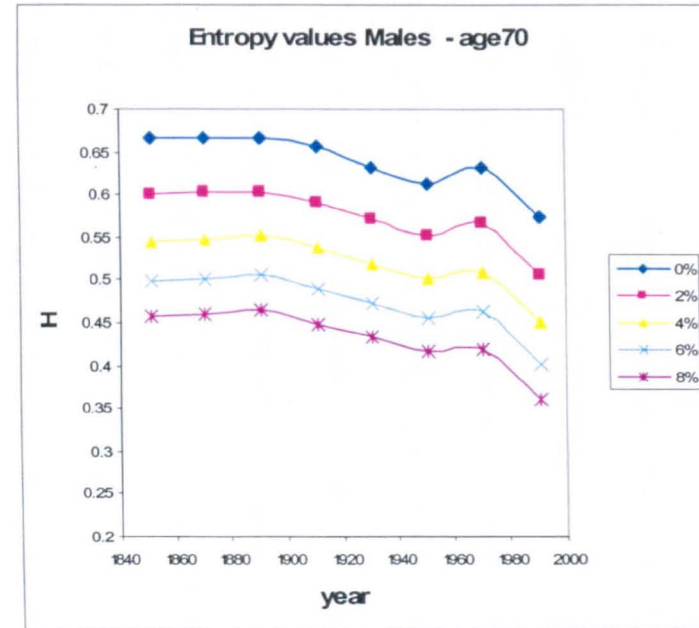
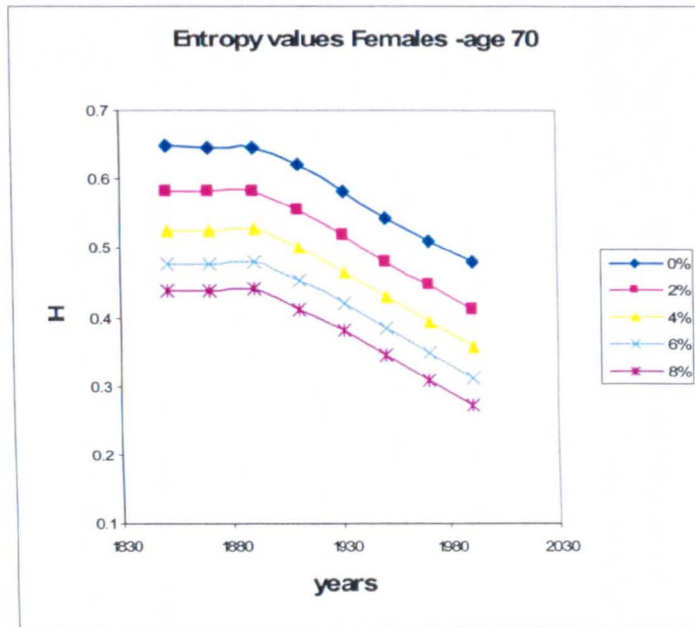
Table 4.7 shows that the values of H are higher than the corresponding ones for males at both age 60 and age 0 and females at age 70, reflecting the higher effect of the change in the force of mortality on the cost of life annuity in the group that experiences higher mortality. The higher is the interest rate the lower is the effect of changes in the force of mortality on the cost of life annuity, hence, the lower the values of H .

At a very high level of mortality, the value of H has increased as mortality improves. Unlike males at age 60 and females at age 60 and 70 the value of H

has increased for all rates of interests at that low mortality level of 1971. One explanation could be that mortality has highly improved in this year for males at age 70 and more as compared to the rest of the cases, which have translated into a stronger effect on the cost of life annuity at that age, hence, the higher value for H . In 1991 mortality has improved again, but the improvement was relatively less than the corresponding one in 1971, thus, the values of H has decreased reflecting a lower effect on the cost of life annuity at this level.

A graphical presentation of the values for H at age 70 for different English life tables and different rates of interest for both males and females is shown in figure 4.3.

Figure 4.3: Entropy values for different English life tables at different rates of interest for both females and males at age 70



We can see that as the rate of interest increases the value of H decreases for both males and females at age 70 at all levels of mortality. Figure 4.3 shows that for both males and females at age 70 at high levels of mortality the value of H increases with the improvements in mortality until it reaches a level when any more improvement is not going to cause a significant effect on the cost of life annuity. Thus, the values of H decrease as mortality improves. At low levels of mortality the values of H has increased for males in 1971 as we mentioned before it might be because there has been a remarkable improvements in mortality for males aged 70 and above at that year. This did not happen for females at age 70 one reason could be that they already experience a lower mortality level as compared to males at age 70 and it is less likely for a substantial improvement in the mortality to happen.

4.4 Numerical results for H using a mathematical model for mortality and allowing for mortality improvements

In this section, the values of the entropy measure H are calculated when mortality is assumed to follow a mathematical model instead of using mortality tables. This will allow us to better examine the different properties of H and how its value is affected by different factors. In section 4.4.1 mortality is assumed to follow the Gompertz model and improvements in mortality are allowed for using a reduction factor. For simplicity it is assumed that the reduction factor depends only on the time t from the base year. The reason behind the choice of the

Gompertz model is that it is a simple model, but at the same time it is reasonable for use at higher ages which are the ages, we are interested in, in this study. In section 4.4.2, mortality is assumed to follow the 1991-1994 mortality tables and a more sophisticated model is used to model mortality improvements, namely the Sithole et al (2000) mortality projection model discussed earlier in chapter 2.

4.4.1 Gompertz law assumption

We assume that the force of mortality is such that it follows the Gompertz's law, and then the base force of mortality at age $x+t$ can then be expressed as

$$\mu_{x+t}^b = \exp[b + c(x + t)] \quad (4.13)$$

where b and c are parameters. If the reduction factor after t years the base year is defined as $RF(t) = \exp[-\alpha t]$ then the new probability for a life aged x to survive t years will be:

$$\begin{aligned} {}_tP_x^* &= \exp\left[-\int_0^t \mu_{x+u}^* du\right] \\ &= \exp\left[-\int_0^t \exp[b + c(x + u) - \alpha u] du\right] \\ &= \exp\left[-\mu_x^b \left(\frac{\exp(c - \alpha) - 1}{(c - \alpha)}\right)\right] \end{aligned} \quad (4.14)$$

Hence,

$$H = \frac{- \int_0^{\infty} (\log_e P_x^*) P_x^* \exp[-\delta t] dt}{\int_0^{\infty} P_x^* \exp[-\delta t] dt}$$

$$= \frac{-\mu_x^b}{c-\alpha} + \frac{\mu_x^b}{c-\alpha} \left[\frac{\int_0^t \exp\left(\frac{-\mu_x^b}{c-\alpha} ((\exp(c-\alpha)t)-1)\right) \exp((c-\alpha-\delta)t) dt}{\int_0^t \exp\left(\frac{-\mu_x^b}{c-\alpha} ((\exp(c-\alpha)t)-1)\right) \exp(-\delta t) dt} \right] \quad (4.15)$$

So $H(b, c, \alpha, \delta)$ is a function of b and c which represent the base mortality table, α which represents the level of mortality change and the force of interest δ . Clearly the range of values that α can take is such that c is greater than α otherwise the probability of survival in 4.14 will be greater than 1. H can be calculated for different levels of b , c , α and δ in order to test the effect of the base table used, the level of mortality improvements and the force of interest on the value of H .

Each of the integrals in (4.15) can be written as an incomplete gamma function, by several changes of variables. The formula and the derivation can be seen in Appendix (1). Alternatively, numerical approximation for integrals can be used to evaluate the integrals so that we obtain numerical values for H for different

levels of c , α and δ . This can be done through standard statistical packages. In this section calculations have been carried out using a software known as Mathematica, which is a computational software popular in both social and natural sciences research.

Calculations for H have been performed for female pensioners at age 60. The base mortality table used is the (1991-1994) mortality table. To perform the calculations we need to have values for μ_{60}^b , c , α and δ . The value for μ_{60}^b can be taken directly from the base mortality table (i.e. 1991-1994 tables), and the Gompertz model can be fitted to estimate c as follows:

The force of mortality is defined as $\mu_{x+t}^b = \exp[b + c(x + t)]$,

where c can be thought of as the slope of the regression line that relates $\ln \mu_{x+t}^b$ with age, x . We have defined the reduction factor at time t to be

$RF(t) = \exp[-\alpha t]$ so that the higher the value of α the lower the RF , hence the lower are the future rates of mortality. When α is positive, the RF will be less than 1, and when it is negative the RF will be more than 1, i.e. mortality rates are increasing. We will use a range of (0.07, -0.07) for the value of α .

The force of interest δ will take values from 0% to 10% in order to test the effect of the change of the rate of interest on the value of H .

The value for μ_{60}^b using the 1991-1994 tables for females is 0.00552155. A regression line has been fitted and the value of the slope obtained, and the estimate for c is 0.085.

A graphical presentation of the values of H when $c=0.085$ for different values of α and δ is given in figure 4.4.

Figure 4.4: Values for H when mortality follows the Gompertz law for different rate of interest and levels of mortality improvements

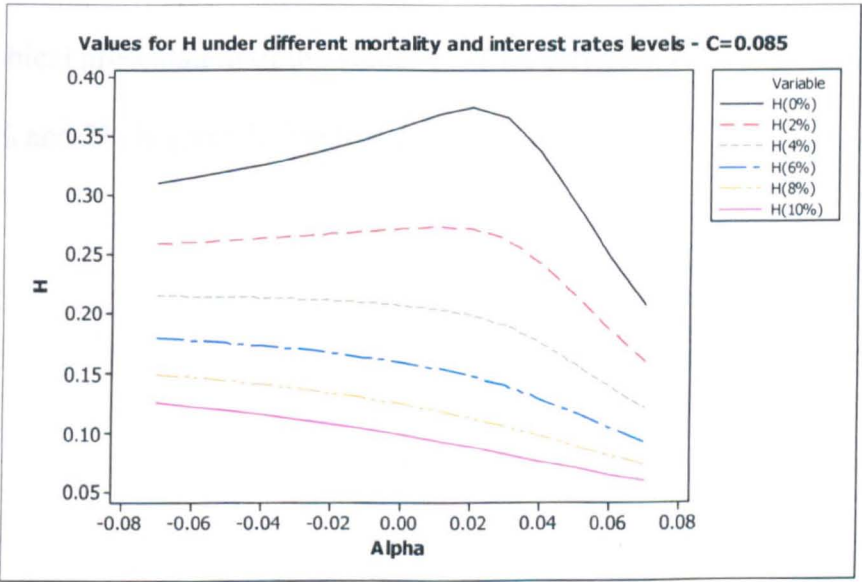


Figure 4.4 shows that the value of H decreases when the force of interest, δ , is increasing, reflecting the decreasing effect of mortality improvement on the cost of life annuities at high levels of interest rates. It can also be seen that at zero or low levels of interest the value of H increases with the improvement in mortality

(as α increases) until a level where any further improvement in mortality does not have a significant effect on the cost of life annuity. Hence, the value of $//$ starts to decrease as mortality improves exhibiting a similar pattern to that of females aged 60 and 70 in section 4.3.2. On the other hand, when the force of interest is high, $//$ decreases as mortality improves all way through.

We can also test the effect on $//$ due to the change of the base mortality table, by calculating $//$ for different values of c . In this chapter we decided to use value of 0.08, 0.085 and 0.09.

A graphical presentation of the values of $//$ for different values of c and α when $\delta = 0\%$ and 2% is given in figure 4.5.

Figure 4.5: Values for H for different values of c and α - mortality follows the Gompertz law

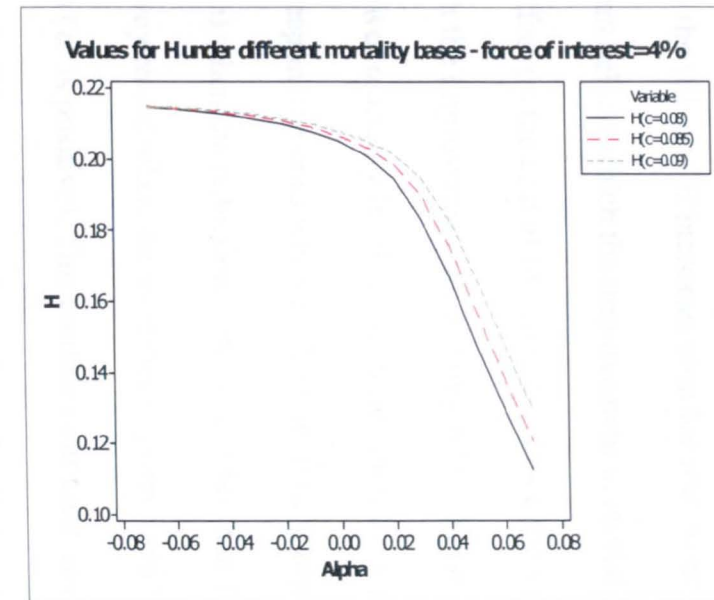
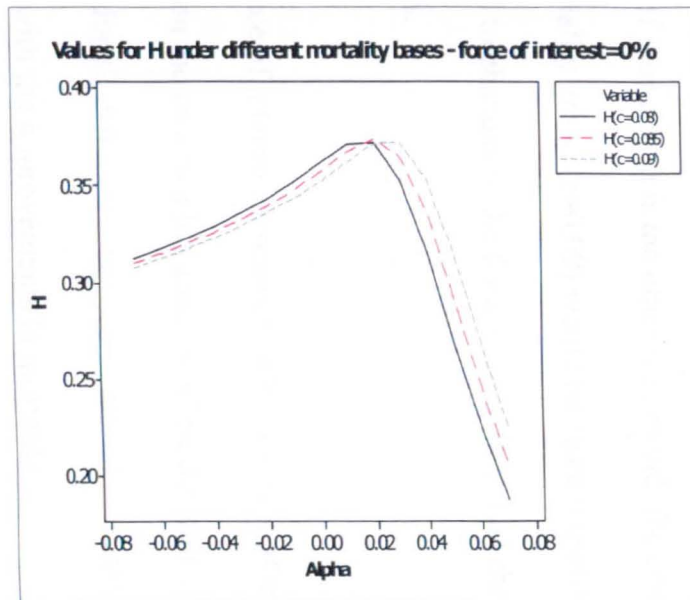


Figure 4.5 shows that for all different values of c when the force of interest is equal to zero the value of H increases with the improvement in mortality until it reaches a point behind which the improvement in mortality does not have a significant effect on the cost of life annuity. Hence the value of H starts to decrease with the improvement in mortality rates. When c is equal to 0.08 (assuming lower mortality level for the base table) the values of H are higher than the corresponding ones when $c=0.09$ (assuming higher mortality level for the base table) when the reduction factor is greater than 1 (α is negative) and it is the other way round when the mortality improves and the reduction factor is less than one (α is positive). This confirms our conclusion before in section 4.3 that the cost of a life annuity for the high mortality group ($c=0.09$) would be less responsive (hence have lower values for H) to changes in the force of mortality than the corresponding one when $c=0.08$ at very high levels of mortality. While at low level of mortality it is the other way round: the cost of a life annuity for the high mortality group ($c=0.09$) would be more responsive (hence have higher values for H) to changes in the force of mortality than the corresponding one when $c=0.08$.

When the force of interest increases to 4% the effect of any change in the force of mortality on the cost of a life annuity is less dramatic, resulting in lower values for H for all different scenarios of mortality base table. Also the value for H decreases with the improvements in mortality.

One advantage of using a mathematical model for the rates of mortality is that it enables us to investigate further the mathematical properties of the derivative of H with respect to the different variables concerned, i.e. with respect to c , α and δ . Hence, this should help us to better understand the behaviour of the entropy measure H . The numerical values of the partial derivative of H with respect to different levels of the improvement factor α are consistent with results that we have obtained so far, which confirms the conclusions that we have reached regarding the relationships between different variables. The expressions for the partial derivatives of H together with the numerical values are given in appendix (2).

4.4.2 Mortality according to the (1991-1994) mortality table for female pensioners with mortality improvements allowed for using Sithole et al (2000)

In this section, the mortality base table is the 1991-1994 mortality table and mortality improvement is allowed for using the Sithole et al (2000) mortality projection model for pensioners mentioned in chapter 2.

Again, the value of H has been calculated for different interest rates and different levels of mortality improvements.

The Sithole et al (2000) reduction factor can be written in the form

$$RF(x, t) = \exp[(-\alpha + \beta x)t]$$

To consider different levels of mortality improvement, we will allow α to vary. The value of α as in Sithole et al (2000) reduction factor is 0.050651, and H has been calculated with values of α over the range from 60% to 140% of the corresponding value of the Sithole et al (2000) reduction factor.

A graphical presentation of the values of H at different rates of interest and for different values of α is given in figure 4.6.

Figure 4.6: Values for H at different rates of interest and mortality improvements allowed for using the Sithole et al (2000) model

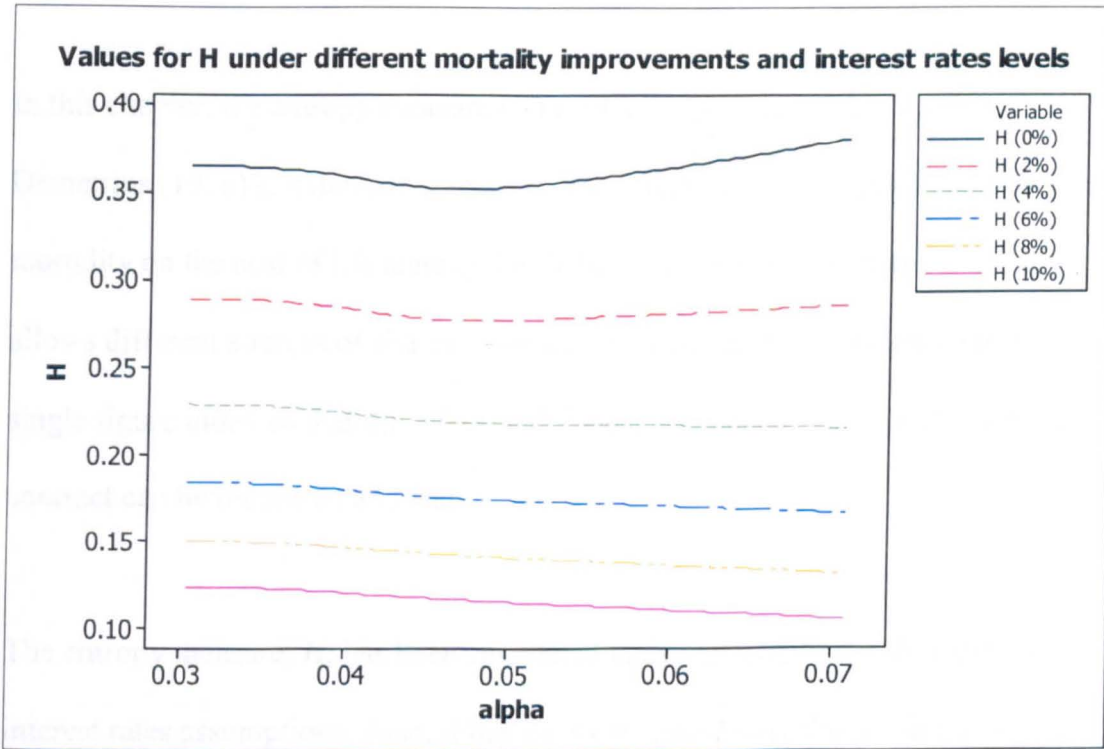


Figure 4.6 shows again that the value of H decreases when the rate of interest increases for all rates of interest. At higher rates of interest the value of H

increases with the increase in the rates of mortality, while at lower rates of interest, the value of H falls with the increase of the rates of mortality until it reaches a point where the increase in mortality rates has a greater effect on the cost of life annuity. At this point, the value of H starts to increase again. We note that the levels of the improvement factor α considered in figure 4.6 lead to very low values of mortality. The pattern of results shown is similar to that we have obtained using the English Life Tables as well as those obtained using the Gompertz law of mortality.

4.5 Summary and remarks

In this chapter, the entropy measure (H) applied in population biology by Demetrius (1976) is extended to measure the effect of any changes in the force of mortality on the cost of life annuity for different interest rate scenarios. This allows different sources of risk in a life annuity contract to be summarized in a single figure index so that the effect of different sources of risk in a life annuity contract can be measured and analysed.

The entropy measure, H , has been calculated under different mortality and interest rates assumptions. Also, it has been calculated over the whole age range (age 0) and at older ages (ages 60 and 70). For all cases, the lower the rate of interest, the higher the value for H . Indicating a higher effect of longevity risk on

the present value of annuity payments. This reflects the importance of longevity risk in the contents of life annuity especially in a low interest environment.

At very high or low levels of mortality, the numerical results for H suggest that the effect of mortality changes on the value of life annuity is of less importance. This reflects the fact that when mortality is already very high or very low, any change (whether an increase or a decrease in the force of mortality) is less likely to have a significant effect on the cost of life annuity. This means that – theoretically - even if mortality continues to improve it will reach a level beyond which any more improvements would not affect the cost of survival benefits by much.

It would be of interest to investigate the mathematical properties of the derivative of H with respect to the different variable concerned in order have a better understanding regarding the way H behaves and being able to answer questions such as, at a given level of interest rate, what would be the level of mortality that is considered too high or too low such that the cost of life annuity is less likely to respond to and further changes in the force of mortality.

The model for H can be extended to reflect the effect of the uncertainty of future returns on the cost of life annuity, by incorporating a model for the interest rate. Also, so far we have used scenario analysis to test the sensitivity of results due to

the change of the main factors affecting H . Using simulation techniques would allow us to consider parameter and model risk for both mortality projection and interest rate models.

It is worth mentioning that there are some limitations related to the use of the entropy value to measure of the effect of change of mortality on the cost of life annuity. Namely, if the survival function is already very rectangular shaped (hence H is close to zero), an expansion of the survival function will have a big impact on the cost of life annuities but this would not be captured by the entropy measure. Also, in the case of complete rectangularization of the survival function, H is going to be zero regardless of the age at which all members of the population die.

Chapter 5

Bayesian Analysis for changes in the cost of life annuity due to longevity risk

5.1 Introduction

In chapter 3 the present values of life annuities have been analyzed using a simulation-based approach. Using the simulated distributions, conclusions regarding the mortality risk in the context of a life annuity portfolio have been drawn. In this chapter we adopt a Bayesian approach that combines the estimation of the parameters of the Sithole et al (2000) mortality projection models and the Lee-Carter projection methodology (both of which have been described in details in chapters 2 and chapter 3) together with the simulation of the annuity cost.

As noted by Congdon (2003) Bayesian methods in econometrics, including applications to linear regression, serial correlation in time series, and simultaneous equations, have been developed since the 1960s with the seminal work of Box and Tiao (1973) and Zellner (1971). Fully Bayesian methods now increasingly offer a more comprehensive and a robust approach to modeling as compared to the classical approach to modeling. Bayesian analysis of data has

been greatly facilitated in the last decade by advances in computing power, which have allowed the wider usage of Bayesian applications excessively in health, social and physical sciences. Although Bayesian statistical methods may be seen as the most convenient method for the implementation and analysis of many models arising in actuarial science, it was not until recently that they have been fully used following the improved scope for estimation via iterative sampling methods and the development of Markov chain Monte Carlo (MCMC) simulation methods like the Metropolis-Hastings algorithm and the Gibbs sampler (Gelfand et al. (1990)).

This Chapter is organized as follows: the Bayesian approach to inference and a review of the basis of Markov Chain Monte Carlo (MCMC) method is discussed in section 2. In section 3 a full Bayesian model is constructed to implement the corresponding MCMC- Bayesian analysis needed to estimate the parameters of the Sithole et al (2000) models. This is then used to forecast future mortality rates and simulate the annuity cost. An approach for measuring the effect of parameter uncertainty (estimation error) is presented and implemented. The same analysis is carried in section 4 using a Bayesian version of the Lee-Carter mortality projection methodology.

5.2 The Bayesian approach to Inference

5.2.1 Bayesian Inference, basis and advantages

The Bayesian method of inference treats all unknown parameters in any statistical model as random variables and assigns a prior distribution to each parameter and then derives their distribution conditional upon the known information.

Suppose we have observed data y and unknown parameters θ . In order to make a probability statement about θ given y , we must begin with a model providing a joint probability distribution for θ and y which can be expressed as the product of the prior $p(\theta)$ and the sampling distribution $p(y|\theta)$. Following Bayes' theorem the posterior density $p(\theta|y)$ is equivalent to:

$$p(\theta|y) \propto p(\theta)p(y|\theta) \quad (5.1)$$

Thus the posterior distribution $p(\theta|y)$ is proportional to the product of the sampling distribution and the prior distribution.

Generally, Bayesian inference methods have a number of advantages. Firstly, they are not dependent on the assumption of asymptotic normality which

underlies classical estimation methods such as maximum likelihood, Thus, under this framework, we can calculate the probability that a parameter lies in a given interval or the probability of a hypothesis about any of the parameters in the model even when the true parameter density is for example skewed or even multi-modal and can't assumed to be normal. Secondly, Bayesian analysis has the advantage of allowing the introduction of extra information that is based on accumulated knowledge through the specification of prior distributions for the parameters.

5.2.2 Markov Chain Monte Carlo (MCMC) and Gibbs Sampling

In Bayesian inference, in order to update knowledge about the parameters it is necessary to take samples from the posterior density. When the prior distribution and the likelihood are conjugate, a simple analytical result provides a method for sampling the unknown parameter. On the other hand in the case of non conjugate priors or when there are many parameters involved, the joint posterior distribution is complex and might not be available in a closed form. In many cases, Bayesian inference requires multi-dimensional integrations of the joint posterior distribution with respect to each of the parameters involved, and such numerical integrations until recently made Bayesian analysis problematic.

MCMC methods provide a unifying framework in which such complex problems can be analyzed. The basic philosophy behind MCMC is to take a Bayesian approach and carry out the necessary numerical integrations using simulation techniques in order to create a stream of simulated values of each quantity of interest.

Consider a vector random variable $\theta=(\theta_1, \dots, \theta_K)$ with a joint distribution $f(\theta_1, \dots, \theta_K)$. Suppose $f(\theta)$ has a complicated and analytically intractable form, and the expected value of some integrable function $h(\theta)$ is sought. Even if this calculation cannot be performed analytically, it is still possible that the probabilistic model associated with $f(\theta)$ may be simple enough to permit independent random draws $\theta^{(t)}, t=1, \dots, n$. If this is the case, then the desired expectation can be approximated using the sample average.

i.e.

$$E[h(\theta)] \approx \frac{1}{n} \sum_{t=1}^n h(\theta^{(t)}) \quad (5.2)$$

This procedure is called Monte Carlo integration. Unfortunately, many complicated models will not readily permit independent random draws. In this case a MCMC method is used to simulate realizations from a Markov chain that is constructed so that its stationary distribution is the posterior distribution. Thus, this Markov chain has $f(\theta)$ as its stationary distribution (see, for example, Gilks

et al 1996). Various algorithms exist for carrying out the required simulations, including the Metropolis-Hastings algorithm that was first described by Hastings (1970). as a generalisation of the Metropolis algorithm developed by Metropolis et al (1953) by allowing jumping rules to be asymmetric which could be useful in increasing the speed of the random walk (see Gelman et al (1995) for more details) One of the simplest MCMC sampling methods is the Gibbs sampler which is in fact a special case of the Metropolis-Hastings algorithm and was devised by Geman and Geman (1984) and subsequently introduced to the statistics literature by Gelfand and Smith (1990). Gibbs sampling draws samples of each parameter in the posterior density, while regarding all other parameters as constants. This can be done as follows: As before, consider a vector random variable $\theta=(\theta_1, ..., \theta_K)$ with a joint distribution $f(\theta_1, ..., \theta_K)$, where each θ_i is a random variable with a marginal distribution $f(\theta_i)$ and $f(\theta_i | \theta_1, ..., \theta_{i-1}, \theta_{i+1}, ..., \theta_K)$ is the full conditional distribution of θ_i given the remaining variables. Given an arbitrary vector of starting values $\theta^{(0)} = (\theta_1^{(0)}, ..., \theta_K^{(0)})$, the first iteration of the Gibbs sampler proceeds by making random draws from the full conditional distributions as follows:

$$\begin{aligned}
\theta_1^{(1)} &\sim f(\theta_1 | \theta_2^{(0)}, \dots, \theta_k^{(0)}) \\
\theta_2^{(1)} &\sim f(\theta_2 | \theta_1^{(1)}, \theta_3^{(0)}, \dots, \theta_k^{(0)}) \\
&\vdots \\
\theta_i^{(1)} &\sim f(\theta_i | \theta_1^{(1)}, \dots, \theta_{i-1}^{(1)}, \theta_{i+1}^{(0)}, \dots, \theta_k^{(0)}) \\
&\vdots \\
\theta_k^{(1)} &\sim f(\theta_k | \theta_1^{(1)}, \theta_2^{(1)}, \dots, \theta_{k-1}^{(1)})
\end{aligned}$$

This completes the first iteration of the algorithm and defines a transition from $\theta^{(0)}$ to $\theta^{(1)} = (\theta_1^{(1)}, \dots, \theta_k^{(1)})$. After t iterations we have $\theta^{(t)} = (\theta_1^{(t)}, \dots, \theta_k^{(t)})$, and the resulting sequence of dependent draws $\theta^{(1)}, \theta^{(2)}, \theta^{(3)}, \dots$ can be then used as described in equation (5.2).

How many iterations or how long a MCMC simulation should be run is a function of the particular application. In any case, the convergence of the MCMC simulation has to be monitored and the first portion of the simulated Markov chain is normally discarded in order to reduce the effect of the starting values.

In order to carry out the MCMC-Bayesian analysis, a specialized software package that performs Bayesian inference using Gibbs sampling known as WINBUGS has been used in this chapter. A useful review of actuarial application of MCMC methods using WINBUGS given by Scollnik (2001).

5.3 Analyzing annuity values in a Bayesian framework using the Sithole et al (2000) mortality model

5.3.1 Methodology

Unlike the case in chapter 3, where the simulated distributions of annuity values were calculated using the maximum likelihood estimates for the Sithole et al (2000) mortality projection models, in this section, a fully Bayesian model is used to estimate the parameters of the Sithole et al (2000) models to forecast future forces of mortality, and hence produce the full distribution of annuity values. We begin by noting that there are a number of sources of error in assessing the effect of mortality improvements on the cost of life annuity, such as the ones in this chapter. These include:

Process error: the actual outcome for a particular portfolio is different from that expected simply because deaths are random events.

Estimation error: the model used is correct, but the true parameters values are different from their estimated values.

Model error: the underlying model used does not reflect the actual improvement in mortality.

Judgement error: error relating to using any subjective judgement in the choice of a distribution, data set used for inference...etc.

Process error is relatively easy to deal with using, for example, simulation methods, and this was one of the sources of error that has been investigated in chapter 3.

Chapter 3 also contained a limited investigation into a combination of estimation error and model error, using an approach often used by actuaries: scenario analysis. Changing the mortality improvement model, as done in chapter 3, could reflect the fact that either the parameter estimates or the model itself do not represent properly mortality changes in the future.

In a Bayesian approach, it is possible to consider estimation error and model error explicitly, and we concentrate on estimation error in this section. A full investigation of model error would postulate a reasonable number of possible models of future mortality rates, assigning to each a prior probability: this is beyond the scope of this chapter.

Estimation error could be accounted for by simulating from the sampling distribution of the parameters of the model of Sitihole et al. It is usual to assume

that the parameter vector has (asymptotically) a multivariate normal distribution whose covariance matrix could be estimated. Given this estimated covariance matrix, it would be possible to simulate values of the parameters from their joint distribution, and then simulate annuity values using these parameter values as in chapter 3. This would lead to an increase in the variability of the distribution of annuity values, reflecting the underlying parameter uncertainty (or estimation error).

In this chapter, Bayesian methods are used as an alternative. We prefer to use the Bayesian approach because, as mentioned earlier, it is not then necessary to make an assumption of normality, and because these methods would also allow us to extend the investigation to include, for example, estimation error. Thus, this section also considers the use of Bayesian methods to assess the effect of parameter uncertainty (estimation error) on the distribution of annuity values.

We start by using a fully Bayesian model to estimate the parameters of the Sithole et al (2000) models, and hence forecast future forces of mortality. Then, a portfolio of persons is considered and we allow the time of death for each person to be a random variable and then calculate the present value of annuity payments for each person and obtain the mean value of the simulated distribution of annuity payments. This can be carried out in the same manner as mentioned earlier in chapter 3, except that it is performed in a Bayesian framework and that

the parameters of the mortality projection model have to be estimated first using Bayesian inference techniques.

In line with the calculations performed in chapter 3, we will consider the CMI data which relates to male and female office pensioners for the period from 1983 to 1996. And for each age for both male and female life office pensioners, we will consider a single life annuity with payments of £1 due at the end of each year and an interest rate of 6% pa.

5.3.2 Prior distributions and estimation of the parameters of Sithole et al (2000) mortality projection models

Within the Bayesian framework, we treat all model parameters as unknowns and we specify prior information via appropriate probability density functions.

Generally speaking, the choice of the prior distributions depends on the opinions and knowledge regarding the parameters involved, so it is impossible to choose a distribution that will necessarily be appropriate under all circumstances. In many cases it is more convenient to use prior distributions that can be guaranteed to play a minimal role in the posterior distribution. Such a prior density is described as vague, diffuse or noninformative. In this section, noninformative priors for the parameters have been used. The most common reason behind using noninformative priors is that there may be very little information on the

parameters, and we would like the prior distribution to reflect this lack of information. Or in other cases, although we have substantial prior information it is considered more objective to present a posterior distribution that reflects only sample information, instead of biasing it with our prior views. Finally, it is frequently very difficult to formulate an appropriate informative prior and using noninformative in this case represents a convenient way out of this difficulty.

Let θ denote the parameter vector $\theta=(\beta_0,\beta_1,\beta_2,\beta_3,\alpha_1,\gamma_{11})$, where $\beta_0,\beta_1,\beta_2,\beta_3,\alpha_1$ and γ_{11} are the 6 parameters of the Sithole et al model with prior probability $p(\theta)$, and let y denote the data vector which, in this case will be the CMI life office pensioners' experience. y will represent the female office pensioners' experience when we estimate the parameters for female pensioners and it will represent the male office pensioners' experience when we estimate the parameters for male pensioners.

Our prior assumption for all of the 6 parameters is a normal distribution with mean 0 and a very large variance², i.e. non-informative priors as we have mentioned above.

The descriptive statistics of the distribution of the estimated parameters values for both male and female office pensioners are shown in table 5.1. Also graphical

² The inverse of the variance is 1.0E-6

presentations of these distributions for both male and female pensioners are shown in figures 5.1 and 5.2 respectively.

Table 5.1: Summary of the descriptive statistics of the parameters of Sithole et al (2000) -male and female pensioners

Male pensioners Model			Female Pensioners model		
Parameter	Estimate (mean)	Standard deviation	Parameter	Estimate (mean)	Standard Deviation
β_0	-2.535	0.003822	β_0	-3.177	0.005547
β_1	1.822	0.01069	β_1	1.838	0.01157
β_2	-0.2639	0.009513	β_2	-0.0263	0.01379
β_3	-0.04892	0.01169	β_3	-0.03614	0.01524
α_1	-0.1257	0.003409	α_1	-0.08306	0.007964
γ_{11}	0.09621	0.00982	γ_{11}	0.05582	0.01725

It can be seen from table 5.1 that the mean values of the distributions obtained using Bayesian inferences are very close to the maximum likelihood estimates obtained by Sithole et al (2000) (see chapter 3) for all parameters.

Figure 5.1: Distributions of the parameter values of the Sithole model –male pensioners

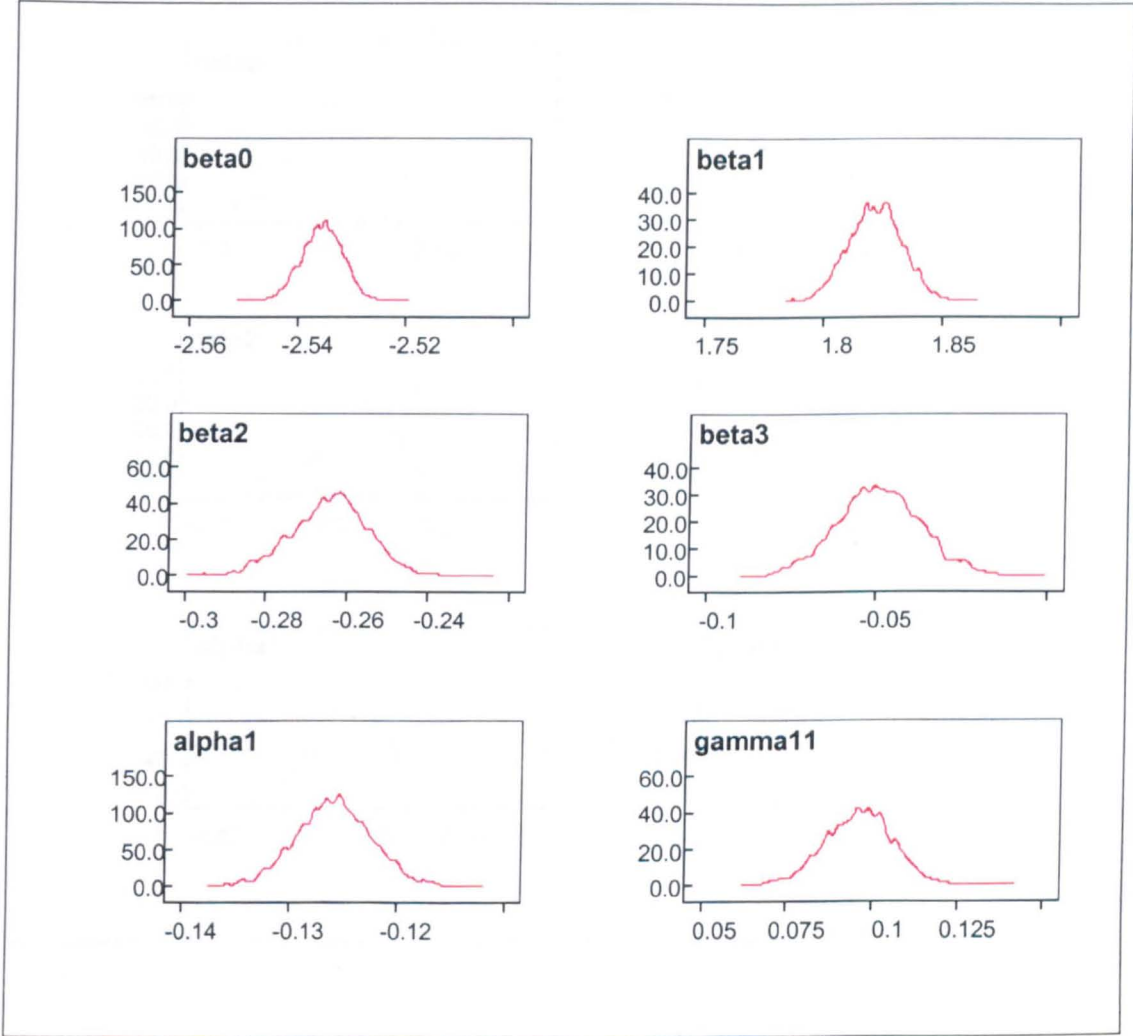
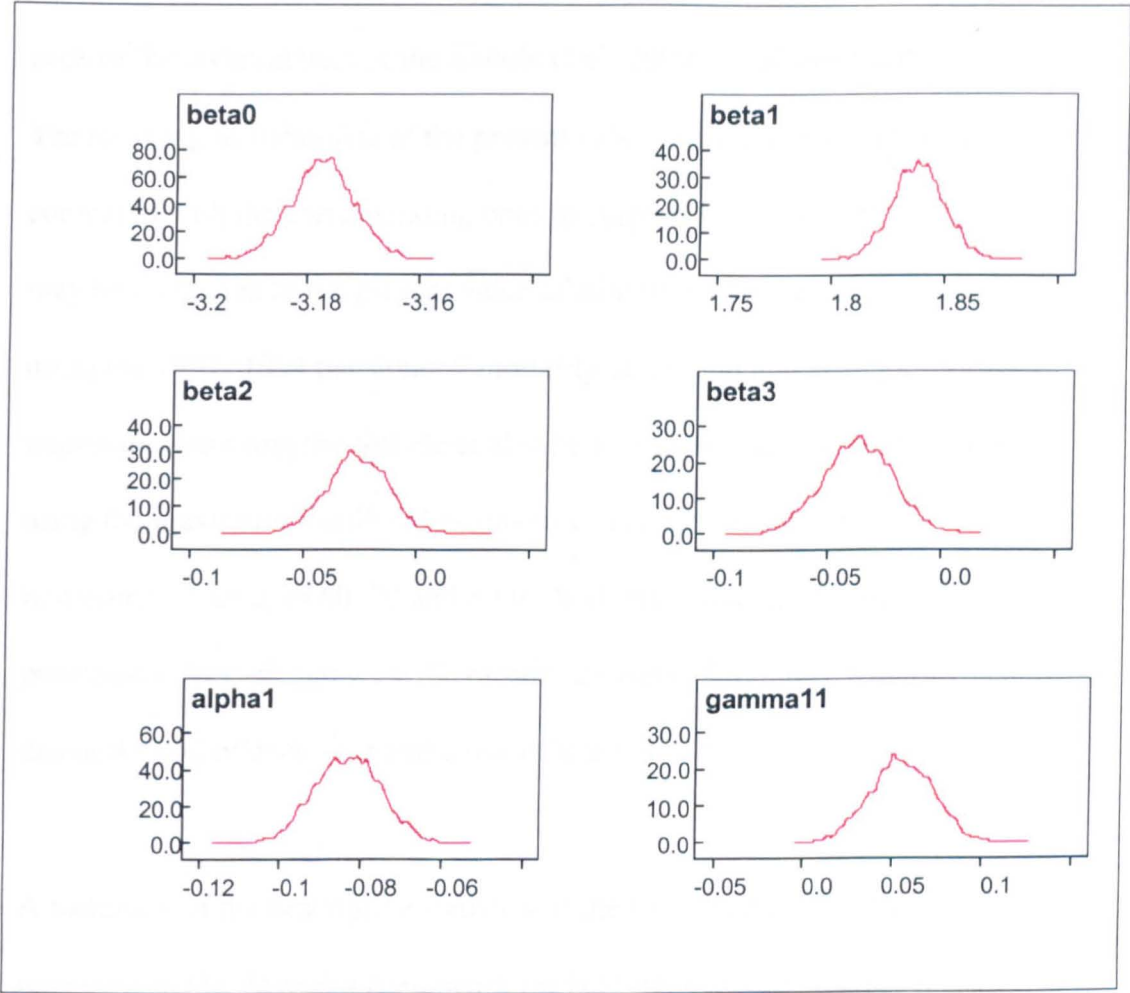


Figure 5.2: Distribution of the parameter values of the Sithole model – female pensioners



5.3.3 Analyzing annuity values

As well as estimating the parameter values, we can include a group of prospective pensioners in the Bayesian model, in order to examine the statistical properties of their future remaining life times in a similar manner to that

described earlier in chapter 3. The only difference is that the simulated distributions of annuity values will be obtained using the whole distribution of each of the 6 parameters of the Sithole et al (2000) projection model.

The resulting distributions of the present value of annuity payments may then be compared with the corresponding ones in chapter 3, and also the mean values may be compared to the present value of annuity payments obtained analytically using the 1991- 1994 pensioners' mortality tables and allowing for mortality improvements using the Sithole et al reduction factors that has been calculated using the maximum likelihood estimates of the parameters. This will be implemented for ages 60, 70 and 80 for both male and female life office pensioners. At each age we will consider a single life annuity with payment of £1 due at the end of each year and a rate of interest of 6%.

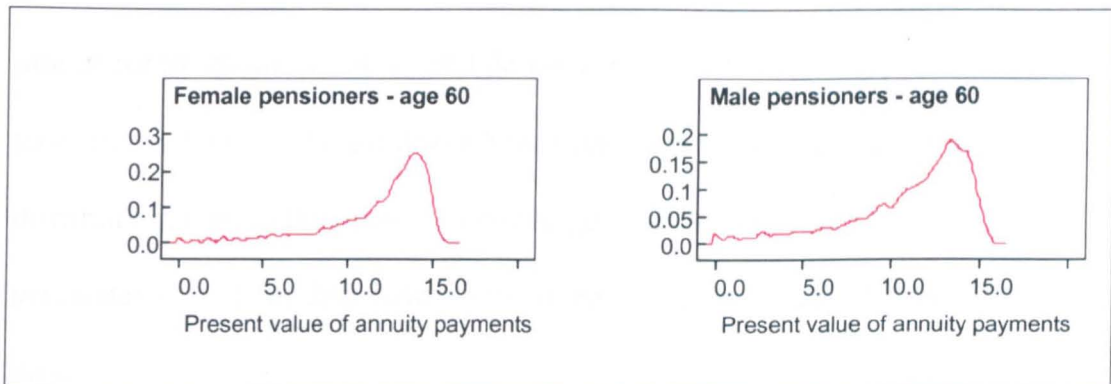
A summary of the descriptive statistics of the simulated distribution of annuity payments in the Bayesian framework for both male and female life office pensioners is shown below in table 5.2. The prediction error is the standard deviation of the distribution of the predicted annuity values, and includes both estimation and process error.

Table 5.2: Summary of the descriptive statistics of the simulated distributions of annuity payments for male and female life office pensioners

Age	Male pensioners Model			Female Pensioners model		
	Mean Value	Prediction Error	Mean Value obtained analytically	Mean Value	Prediction Error	Mean Value obtained analytically
60	11.000	3.508	10.974	11.800	3.161	11.768
70	7.960	3.718	7.882	9.087	3.508	8.969
80	5.011	3.230	4.946	5.940	3.302	5.970

It can be seen from table 5.2 that the mean values of the distributions are close to the corresponding analytical values as well as being close to the corresponding values that have been obtained in chapter 3 for all ages examined for both genders (see basis B results in tables 3.4 and 3.5). It is also clear that the prediction error, as a proportion of the mean, increases dramatically with age for both males and females, reflecting a higher level of variability as age increases. For males it is 31.9%, 46.7% and 64.5% for ages 60, 70 and 80 respectively, and for females it is 26.6%, 38.6% and 55.6% for ages 60, 70 and 80 respectively. A graphical presentation of the distributions of annuity values for both male and female pensioners at age 60 is shown in figures 5.3.

Figure 5.3: Distribution of annuity values for male and female pensioners age 60



The distributions in figure 5.3 can be compared to the corresponding figures in chapter 3 (see basis B in figures 3.1 and 3.2). The distributions in figure 5.3 generally show similar characteristics to those in chapter 3, namely, the distribution of annuity payments for male pensioners is less skewed and more spread than the corresponding one for female pensioners.

5.3.4 Convergence diagnostics

Monitoring and assessing the convergence of the MCMC simulation is an important part of any MCMC based analysis and it requires considerable care as it is very difficult to say conclusively that a chain has converged, although easier to diagnose when it definitely has not converged.

A MCMC simulation converges when its output, from that point on, comes from the true stationary distribution of the Markov chain, or, equivalently, when inferences for quantities of interest do not depend on the starting point of the simulations. Once sufficient draws have been taken to summarize the posterior distribution, then, if the model has converged, further samples from a parameter's posterior distribution should not influence the calculation of the mean.

Once we are satisfied that the simulation has converged, the first portion of the simulated Markov chain is discarded in order to reduce the effect of the starting values. These discarded or "burn-in" iterations are the pre-convergence iterations. The subsequent updates give the samples from the posterior distributions. For the Sithole et al (2000) mortality projection model the number of updates that the simulation has to be run is such that the Monte Carlo error for each parameter is less than 5% of the sample standard deviation. The number of burn-in iterations was 1000 for all cases, and the total number of updates afterwards was 20,000.

Graphical techniques are commonly used to check the convergence of MCMC simulations. One way to do that is by examining the trace and history plots of the sample values versus iteration number to look for evidence of when the simulation appears to have stabilised. Also several chains with different starting

values can be run simultaneously. In this case, we can be reasonably confident that convergence has been achieved if all of the chains appear to be overlapping on one another.

Figure 5.4 gives the history plot for the 6 parameters of the Sithole et al (2000) mortality projection model for both male and female pensioners after the burn in iterations, while figure 5.5 gives the corresponding trace plot but using two different sets of initial values (i.e. using two chains).

Figure 5.4: History plot for the parameters of the Sithole et al (2000) mortality projection model

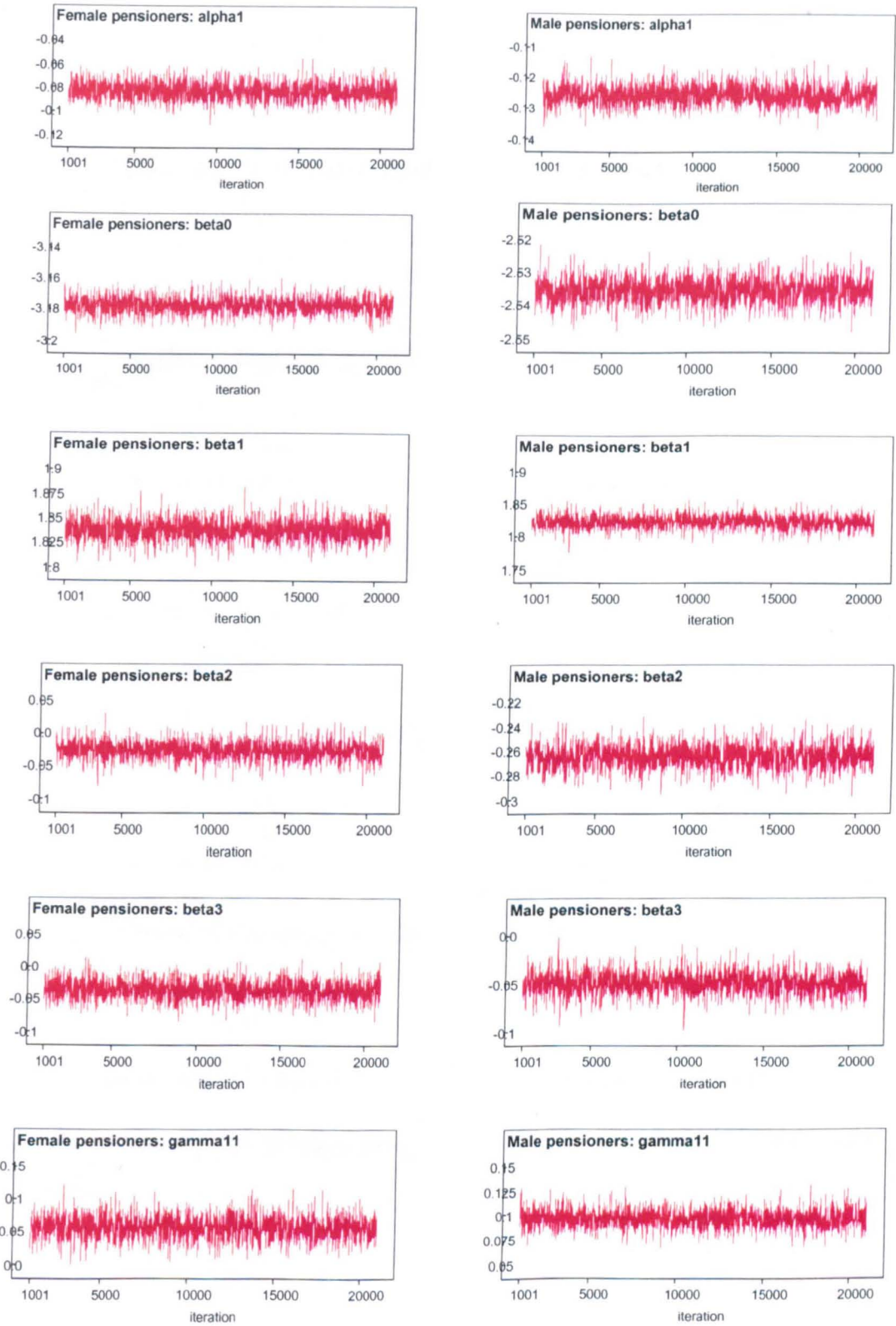
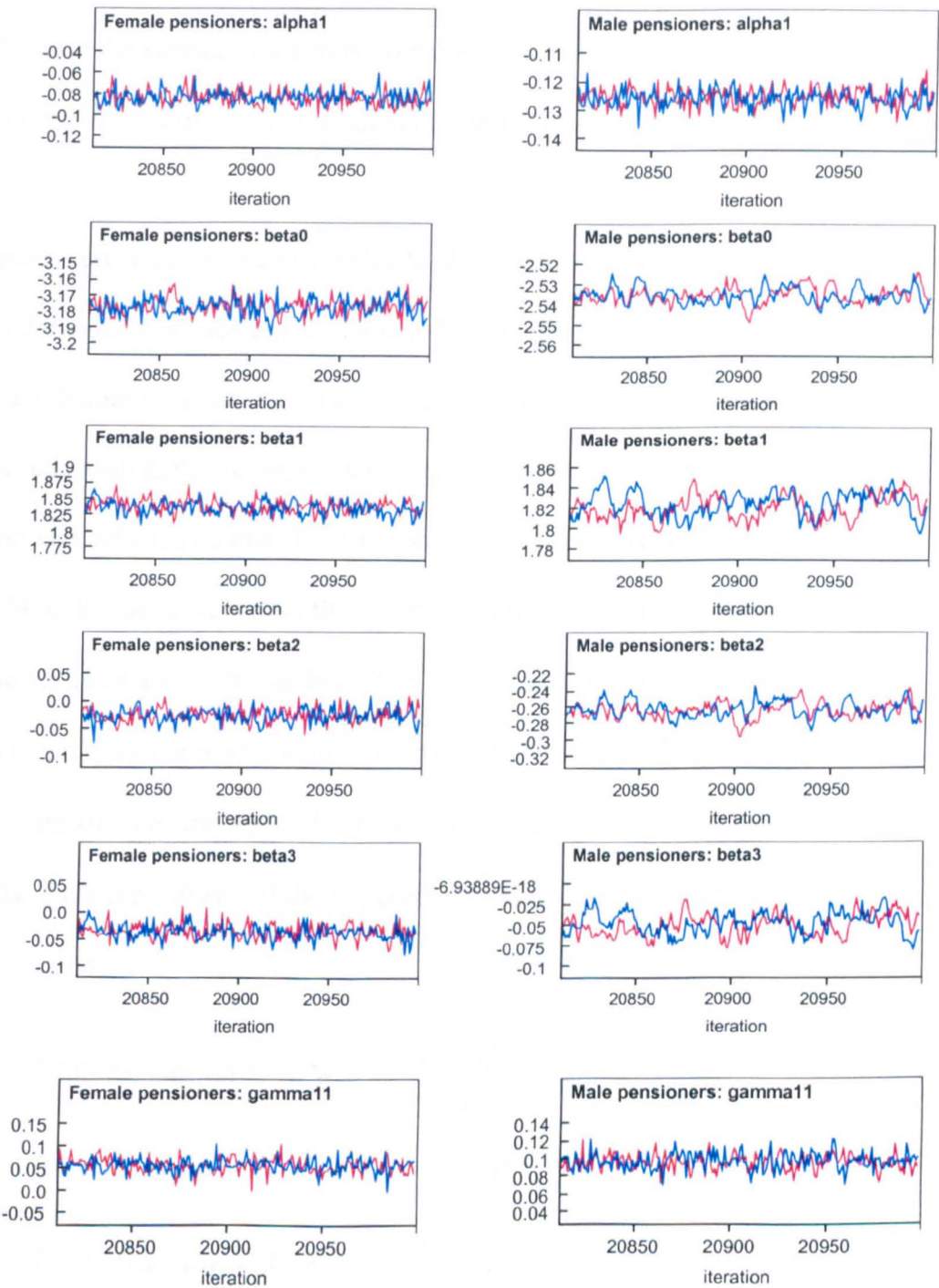


Figure 5.5: Trace plot for the parameters of the Sithole et al (2000) mortality projection model



From figure 5.4 we can see that, after the burn in iterations, the simulation appears to have stabilised for all of the parameters indicating that convergence has occurred. This is confirmed by using two chains of starting values (figure 5.5) where the simulations appear to exhibit the same behaviour through the iterations irrespective of the initial values used.

Alternatively, convergence of multiple chains can also be assessed using more formal diagnostics such as the Gelman-Rubin (1992) convergence statistic (the Gelman-Rubin scale reduction factors), which is included in WINBUGS. The scale reduction factors compare variations in the sampled parameter values within and between chains. The intuition is that the behavior of all of the chains should be the same, such that the variance within the chains should be the same as the variance across the chains. Thus, the statistic involves two steps:

- 1) Estimate the model with a variety of different initial values for the parameters and iterate for n iterations after the burn-in iterations.
- 2) Take the n draws of the m parameters and calculate the following statistics:

$$\text{Within chain variance } W = \frac{1}{m(n-1)} \sum_{j=1}^m \sum_{i=1}^n (\theta_j^i - \bar{\theta}_j)^2$$

$$\text{Between chain variance } B = \frac{n}{m-1} \sum_{j=1}^m (\bar{\theta}_j - \bar{\theta})^2$$

$$\text{Estimated variance } \hat{V}(\theta) = \left(1 - \frac{1}{n}\right)W + \frac{1}{n}B$$

$$\text{The Gelman - Rubin Statistic } \sqrt{R} = \sqrt{\frac{\hat{V}(\theta)}{W}} \quad (5.3)$$

where

θ_j^i is the i th sample of the j th chain.

$\bar{\theta}_j$ is the mean of the j th chain.

$\bar{\theta}$ is the overall mean (the average of the chain means $\bar{\theta}_j$)

Before convergence, W underestimates the total posterior variance in θ because it has not fully explored the target distribution (i.e. the distribution obtained when convergence is achieved). $V(\theta)$ on the other hand overestimates the variance in θ because the starting points are over-dispersed relative to the target density. Once convergence is reached, W and $V(\theta)$ should be almost equivalent because variation within the chains and variations between the chains should coincide since in this case the output from all chains is indistinguishable, so that R should approximately be equal to one.

For a detailed description of the Gelman and Rubin method as well as other convergence diagnostics, a useful review can be found in Cowles and Carlin (1996).

The Gelman-Rubin (1992) convergence statistic plotted against iteration number for the Sithole et al (2000) parameters for both male and female pensioners is given in figure (5.6).

Figure 5.6: The Gelman-Rubin convergence statistic for the parameters of the Sithole et al (2000) mortality projection model

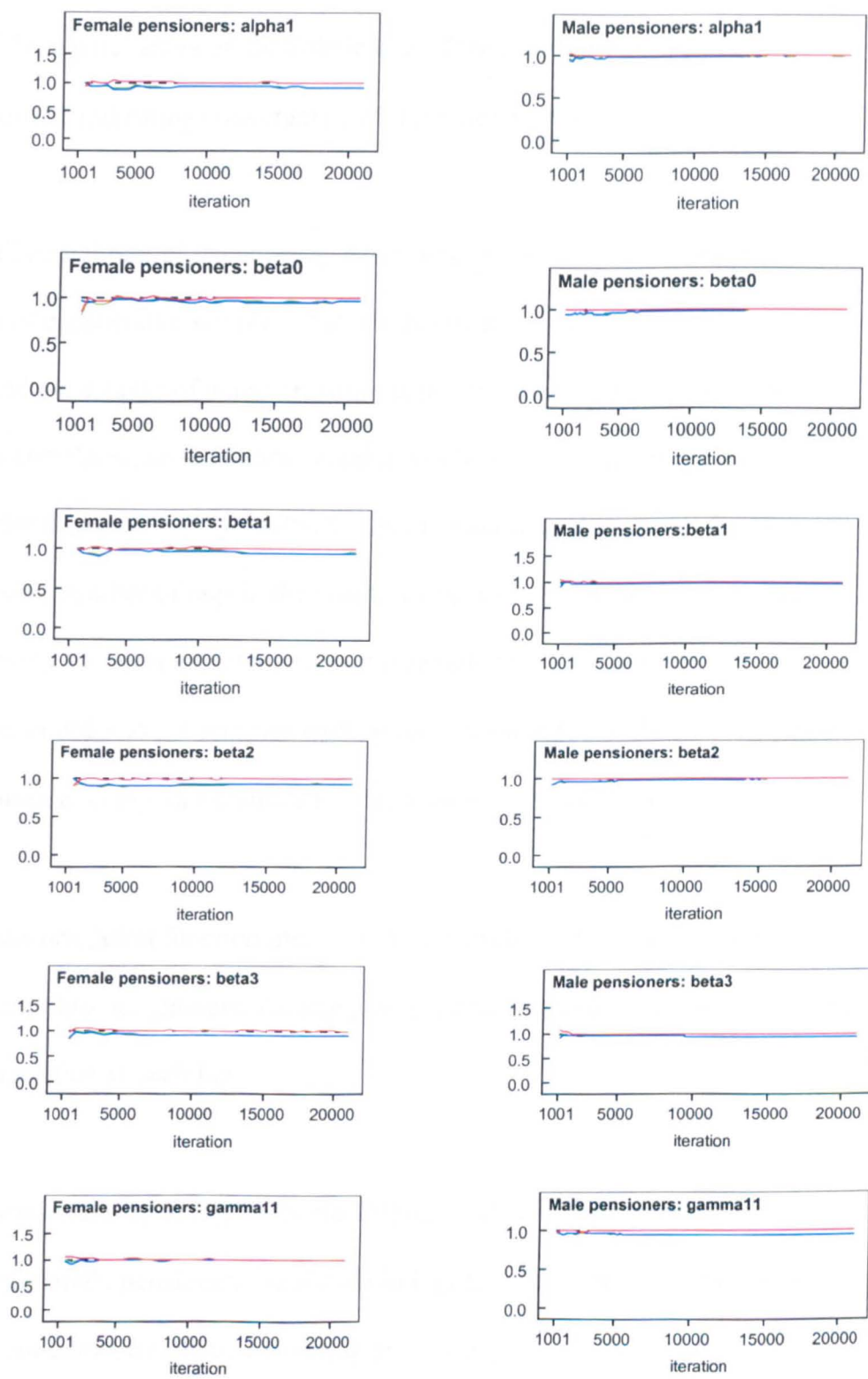


Figure 5.6 shows that the value of the Gelman-Rubin statistic is equal to one for all of the 6 parameters of the Sithole et al (2000) for both male and female pensioners, indicating convergence of the different chains.

As an integral part of the convergence check, the autocorrelations between the draws of consecutive samples of all the parameters involved need also to be checked. One cause of autocorrelation is that the parameters in the model may be highly correlated, so the Gibbs sampler will be slow to explore the entire posterior distribution. Typically, the level of autocorrelation will decline with an increasing number of lags in the chain, i.e. instead of considering a realization from every iteration we retain every n th sample to reduce autocorrelation. In this section, in order to get samples with no significant autocorrelations, only one realization in every 10 iterations has been taken from each parameter.

The autocorrelation function measures how correlated the values in the chain with their close neighbours. An independent chain will have approximately zero autocorrelation at each lag.

Autocorrelations up to lag 50 for the Sithole et al (2000) parameters for both male and female pensioners are shown in Figure 5.7 indicating that there is no significant autocorrelations remaining in the samples.

5.3.5 Measuring the effect of parameter uncertainty

In section 5.3.3, the distribution of the present value of annuity payments was calculated, using parameter values that have been estimated within the model itself. This means that, unlike the case in chapter 3, parameter uncertainty has been accounted for, rather than using fixed values in the simulation that are the point estimates of the parameters.

As a measure of parameter uncertainty, we can compare the results from the Bayesian model with the simulation results when the values of the parameters are fixed at the maximum likelihood estimates obtained by Sithole et al (2000). The prediction error of the simulated distribution of annuity payments in a Bayesian framework for both male and female life office pensioners can then be divided into process and estimation error as shown below in table 5.3.

Table 5.3: Estimation error of the mean value of the simulated distribution of annuity payments

Age	Male Pensioners Model			Female Pensioners model		
	Process Error (1)	Estimation Error (2) [*]	Prediction Error (3)	Process Error (1)	Estimation Error (2) [*]	Prediction Error (3)
60	3.431	0.731	3.508	3.106	0.587	3.161
70	3.700	0.365	3.718	3.500	0.237	3.508
80	3.229	0.080	3.230	3.289	0.293	3.302

Note^{*} : (2) = $\sqrt{(3)^2 - (1)^2}$

It can be seen from table 5.3 that in this case estimation error is not very significant because of the nature of the data (a large data set) while the process error is clearly more important. Model error has not been considered, but we would expect it to be significant.

5.4 Analyzing annuity values in a Bayesian framework using the Lee-Carter (1992) mortality projection model

In this section, the Lee-Carter mortality projection methodology that has been described in chapter 2 will be extended using the Poisson log-bilinear model of Brouhns et al (2002) and then fitted in a Bayesian framework to perform a similar analysis to the one in section 3 of this chapter.

As mentioned in chapter 2, the Lee-Carter (1992) model is a simple model that describes the change in mortality as a function of a single time index. The model represents the log of the age specific central rate of mortality as the sum of an age-specific component that is independent of time, and the product of a time varying parameter reflecting the general level of mortality (also known as mortality index) and an additional age-specific component that describes how rapidly or slowly mortality at each age varies due to changes in the mortality index. The main statistical tool for fitting the model is least squares estimation via singular value decomposition (SVD) of the log age specific observed forces of mortality. However, using least squares SVD implies that the error terms in

the model (equation 2.16) are assumed to have a homoscedastic additive Gaussian structure. This is not a reasonable assumption as the logarithm of the observed force of mortality is much more variable at older ages than at younger ages because of the much smaller number of deaths at older ages. Thus, Renshaw and Haberman (2003 a,b) and Brouhns et al (2002) have implemented alternative approaches based on a heteroscedastic Poisson non-additive error structure. Renshaw and Haberman (2003 a,b) have also presented a re-interpretation of the model underpinning the Lee-Carter methodology for forecasting mortality where a parallel methodology based on generalized linear modeling is introduced and the two methods are compared in terms of structure and assumptions. On the other hand, Brouhns et al (2002) have implemented a suggestion of Alho (2000), which enables the bilinear structure of the Lee-Carter model to be fitted by optimizing the Poisson likelihood. The structure of Brouhns et al (2002) has been employed to forecast mortality for French male population using a Bayesian log bilinear Poisson regression model by Czado et al (2005). We seek to perform a similar analysis with a different underlying Bayesian structure in terms of the prior distributions assumed for different parameters. Also, the analysis is extended to incorporate the calculations of annuity values.

We continue with a description of the Poisson log-bilinear model of Brouhns et al (2002), which will be implemented in a Bayesian framework to estimate the

parameters of Lee-Carter (1992) projection model, and hence calculate annuity values in a similar way as in section 5.3.

For a discussion of various extensions, applications and problems associated with the classical Lee-Carter (1992) mortality projection model, the reader is referred to Lee (2000).

5.4.1 Poisson log-bilinear model of Brouhns et al (2002)

Brillinger (1986) and Alho (2000) showed that the Poisson distribution is a good candidate to model the number of deaths at different ages. The approach of Brouhns et al (2002) describes an extension to the Lee-Carter approach whereby the number of deaths D_{xt} is modelled as a Poisson random variable, such that:

$$D_{xt} \sim \text{Poisson}(E_{xt}\mu_{xt}) \quad (5.4)$$

Where E_{xt} is the exposure-to-risk and μ_{xt} is the force of mortality. The function $\mu_{x,t}$ is then modelled as:

$$\mu_{x,t} = \exp(\alpha_x + \beta_x k_t). \quad (5.5)$$

Hence the force of mortality is assumed to have the log-bilinear form:

$$\ln(\mu_{x,t}) = \alpha_x + \beta_x k_t. \quad (5.6)$$

Note that the meaning of the parameters α_x , β_x and k_t are essentially as defined in the classical Lee-Carter model (2.16) (see chapter 2), and that only the random part of the model has been modified. The parameters hence are estimated by maximising the log-likelihood. The time series part of the Lee-Carter methodology is not modified and is thus used to forecast k_t .

The Poisson likelihood function can then be used in a Bayesian modelling context. In line with what has been done in section 5.3, all of the unknown parameters α_x , β_x and κ_t are treated as random variables and a prior probability density will be assigned to each of the parameters. Then their posterior distribution can be derived, conditional upon the known information, which in this case is the numbers of deaths D_{xt} and the exposure-to-risk E_{xt} .

5.4.2 Model and prior distributions

5.4.2.1 Likelihood function

Using the Poisson log-bilinear model we can see from equations (5.4) and (5.5) - and as noted by Czado et al (2005) - that the likelihood function of the three parameters α , β and κ is

$$L(\alpha, \beta, \kappa) = \prod_x \prod_t \frac{\exp(-E_{xt} \exp(\alpha_x + \beta_x \kappa_t)) (E_{xt} \exp(\alpha_x + \beta_x \kappa_t))^{D_{xt}}}{D_{xt}!} \quad (5.7)$$

$$\propto \prod_x \prod_t \exp(-E_{xt} \exp(\alpha_x + \beta_x \kappa_t) + D_{xt}(\alpha_x + \beta_x \kappa_t))$$

where α , β and κ are three vectors with number of elements equal to the number of age groups in the case of α and β , and equal to the number of calendar years t for κ .

5.4.2.2 Prior distributions for the parameters

We start first by specifying a distribution for α . We recall from chapter 2 that α_x describes the average shape of the age profile over time and that it represents the logarithm of the geometric mean of the crude mortality rates, averaged over all calendar years t , for each age. i.e. $\hat{\alpha}_x = \log \left\{ \prod_{t=1}^n \hat{m}_{xt}^\gamma \right\}$. So it is appropriate to

assign a prior density function that will guarantee that the values for α at each age x are always negative, and this can be done as follows:

We define a_x such that; $\alpha_x = \log_e(a_x)$ and we assign a Gamma probability density function for a_x . i.e.

$$a_x \sim \text{Gamma}(m_\alpha, \sigma_\alpha^2)$$

where the values assigned to m_α and σ_α^2 are such that the prior distribution is noninformative, i.e. has a large variance.

As for β_x , which represents the age-specific pattern of mortality change, and indicates the sensitivity of the logarithm of the force mortality at age x to variations in the time index. The prior assumption that has been used for β is a noninformative normal distribution with mean zero and a large variance:

$$\beta_x \sim \text{Normal}(0, \sigma_\beta^2)$$

This means that we start with the assumption that there are no mortality improvements that will occur for the population under the study, but the fact that the prior distribution is noninformative means that the data of the population under the study will be allowed to transform the prior distribution if improvements do occur.

The structure that has been used for the prior assumptions regarding the time index κ is as follows:

$$\begin{aligned}\kappa_t &\sim \text{Normal}(m_t, \sigma_\kappa^2) \\ m_t &\sim \text{Normal}((m_{t-1} + h_t), \sigma_m^2) \\ h_t &\sim \text{Normal}(h_{t-1}, \sigma_h^2)\end{aligned}$$

So the κ 's are connected to each other through their mean values, as according to this structure the mean value of κ_t (m_t) is a function of the mean value of κ_{t-1} (m_{t-1}) plus a drift (h_t). This structure will allow the drift term to change over time instead of assuming it is a constant value. Also, it allows a great flexibility to control how close are the values of the drift terms (h_t) and the mean values (m_t) to each other. This last feature can be done through the values or the distributions assigned to the hyper-parameters σ_κ^2 , σ_m^2 and σ_h^2 . The model for κ_t is still assumed to be linear, but with the potential advantage of changing the slope at different periods to reflect higher or lower mortality improvements over time which is more reasonable than assuming that the pace of improvement is going to be the same all through¹.

The Gamma distribution is used as the natural prior distribution for the hyper-parameters σ_κ^2 , σ_m^2 , σ_h^2 , σ_β^2 and σ_γ^2 in order to make sure that all the variances have a positive value.

¹ Renshaw and Haberman (2003a) have shown that, when fitting the Lee-Carter (1972) model to UK data, for males, the pattern of the estimated κ_t comprises two linear segments, binged in the mid-1970s.

5.4.3 Estimation of the Lee-Carter (1992) parameters

Three different sets of data have been considered for estimating the parameters of the Lee-Carter (1992) mortality projection model. In each case, the data contain the number of deaths together with the corresponding exposure, cross classified by individual calendar year and by age as follows:

Case A: England and Wales Male Mortality Experience, 1950-1998, inclusive, with age grouped and classified as $\{<1, 1-4, 5-9, 10-14, \dots, 80-84, 85+\}$.

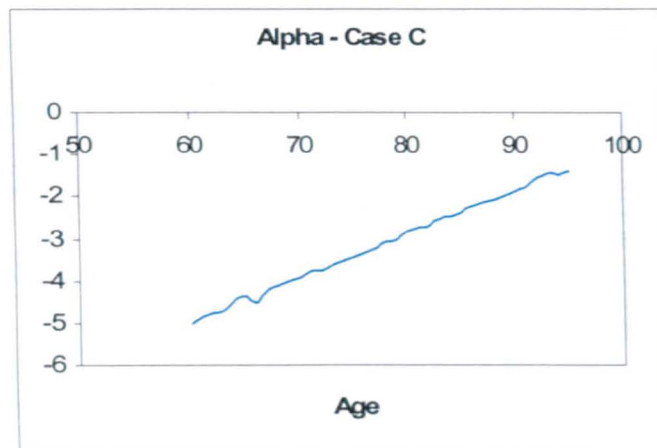
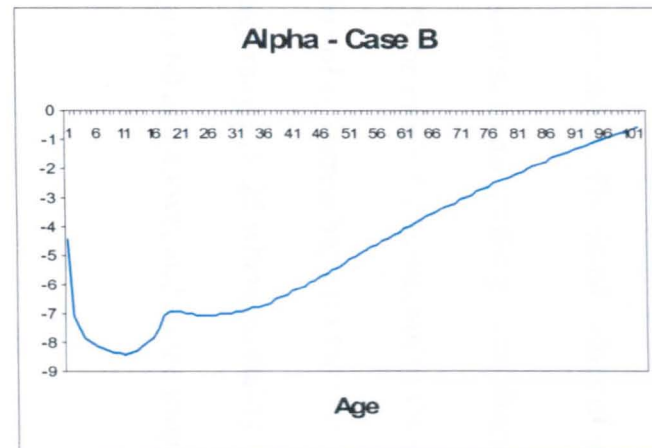
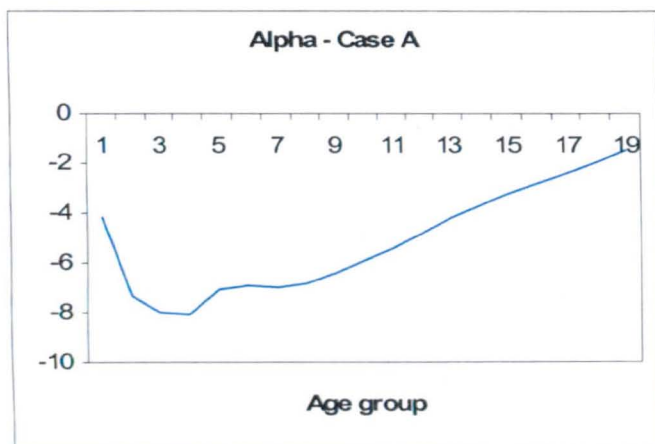
Case B: UK male mortality experience, 1961-2003 inclusive, with age classified by individual year, 0-101 inclusive.

Case C: CMI data for female life office pensioners, 1983-1996 inclusive, with age classified by individual year, 60-95 inclusive.

Using the likelihood function and the prior distributions mentioned in section 5.4.2 the Lee-Carter model has been fitted to the data in the three cases A, B and C. The parameters of the Lee-Carter (1992) are estimated for the three cases but the analysis for the third case (Case C) only is extended to calculate annuity values, so that the results can be compared to the corresponding ones in section 5.3.

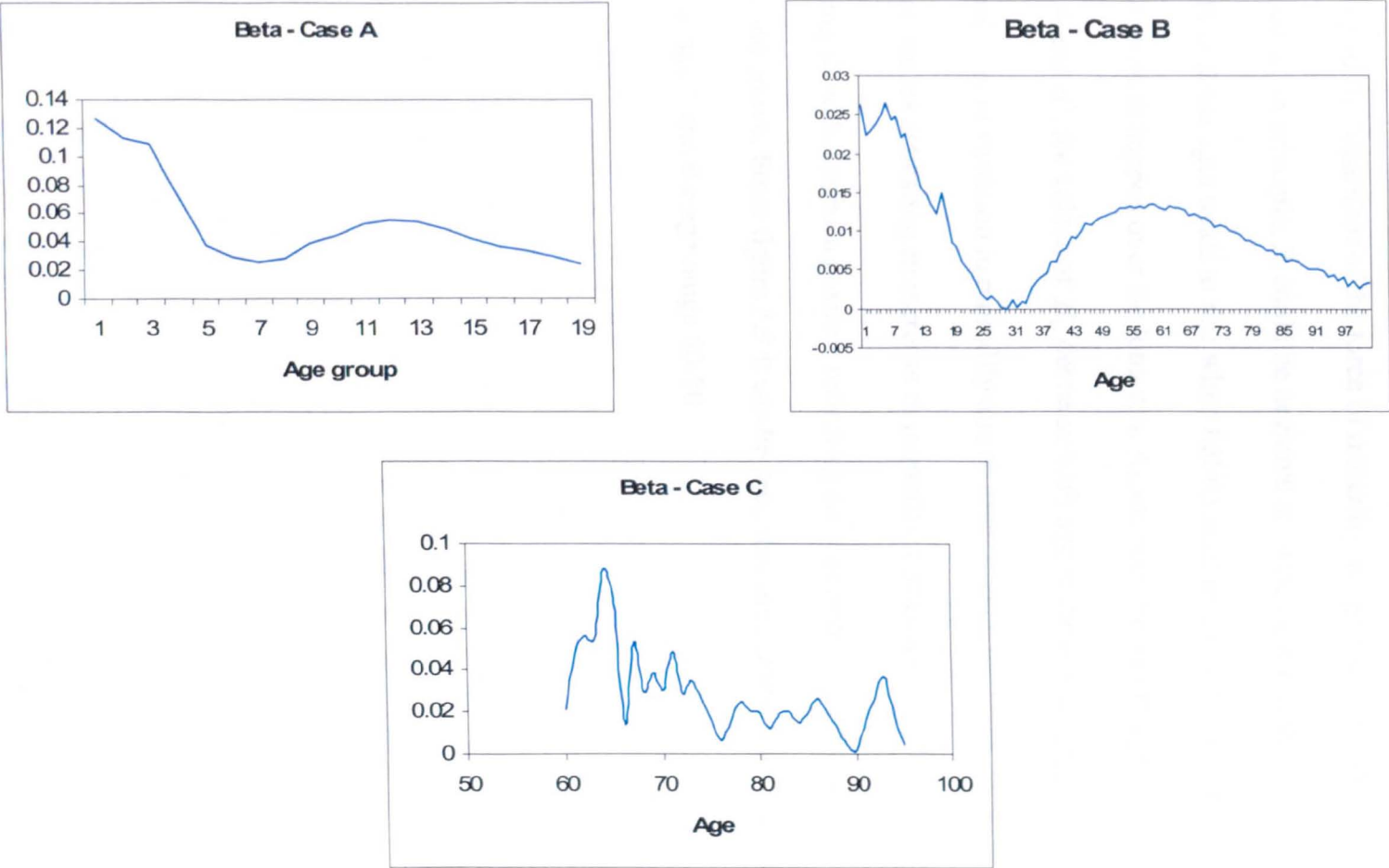
The values of α_x , β_x and κ , for each case, are displayed in figures 5.8, 5.9 and 5.10 respectively.

Figure 5.8: Estimates for α_x - different cases



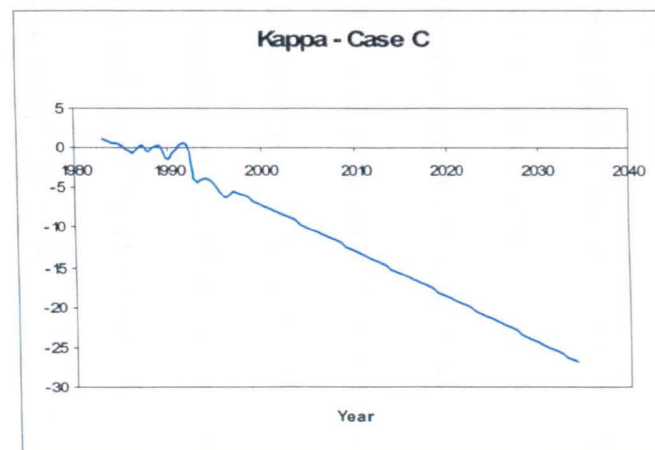
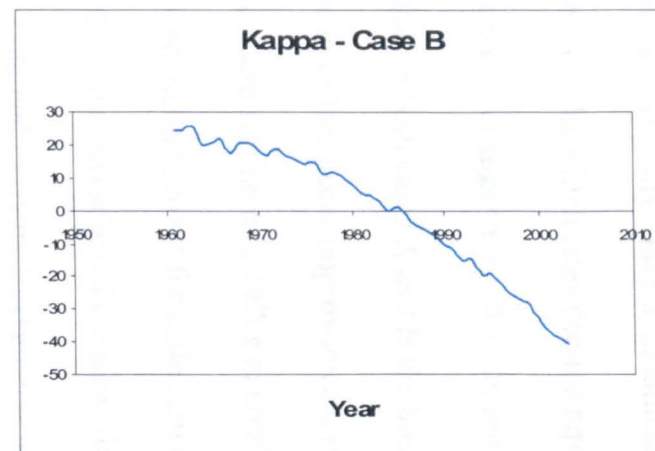
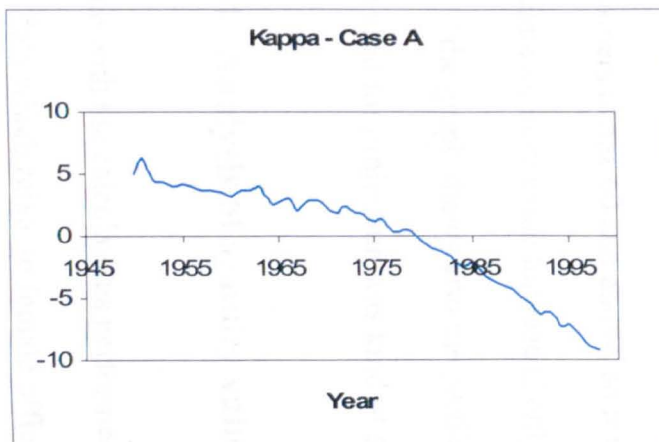
The pattern in the estimates of α_x is almost identical for cases A and B, as in both cases the data were over the whole age range, while in case C the data were for age 60 and above. The fitted values of α_x represent the average of $\ln \mu_{x,t}$ over time t so that $\exp \alpha_x$ is the general shape of the mortality schedule. So, for cases A and B the graph for α_x resembles the general shape of the mortality curve with the value of α_x increasing with the increase in age except at very young ages and at the age range 18-25 where mortality is known to be higher. Case C involves only ages 60 and above, and so the graph of α_x is increasing with age.

Figure 5.9: Estimates for β_x - different cases



β_x represents the age-specific patterns of mortality change. It indicates the sensitivity of the logarithm of the force of mortality at age x to variations in the time index κ_t . In principle, β_x could be negative at some ages x , indicating that mortality at those ages tends to rise when falling at other ages. In practice, this does not seem to happen over the long run. Again over the whole age range (Cases A and B), the values of β_x decrease with age at the beginning reflecting less sensitivity to variation in mortality due to time. It even becomes negative at age 30 (Case B) indicating an increase in mortality. For case C, β_x has a decreasing trend but remain positive reflecting an improvement in mortality for ages 60 and above. From figure 5.9 it can be seen that the highest improvements happen at age 1 and the age range 60-70.

Figure 5.10: Estimates for κ_t - different cases



κ_t represents the time trend. The actual forces of mortality change according to an overall mortality index κ_t modulated by an age response β_x . The shape of the β_x profile tells which rates decline rapidly and which slowly over time in response of change in κ_t . We can see that κ_t decreases over time linearly. We can see that in both cases A and B the linear function of kappa over time changes slope to reflect faster improvements in mortality after the change in the slope somewhere around mid 1970s to confirm what has been observed by Renshaw and Haberman (2003a) for male experience in the UK. Renshaw and Haberman (2003b) have investigated the same data sets in cases A and B, the fitted values of κ_t obtained in figure 5.10 are more or less the same as those of Renshaw and Haberman (2003b). Both cases A and B, deal with population data over the whole age range while in case C the data set consists of female life office pensioners at age 60 and above, so it was natural to extend the analysis to calculate annuity value for case C only as it is the most relevant case. Hence, in case C the graph shows also the predicted values of kappa. The actual period that have used for projection was kind of short (14 years).

5.4.4 Analysis of annuity values

In line with the calculations performed in the section 5.3, we will consider the CMI data which relate to female office pensioners for the period from 1983 to 1996 (case C) in order to analyze the cost of life annuity. Again we will consider

a single life annuity at ages 60, 70 and 80 with payments of £1 due at the end of each year and a rate of interest of 6 percent.

A summary of the descriptive statistics of the simulated distributions of annuity payments for female pensioners using a Bayesian version of the Lee-Carter (1992) mortality projection model is shown below in table 5.4.

Table 5.4: Summary of the descriptive statistics of the simulated distributions of annuity payments for female pensioners using the Lee-Carter (1992) projection model

Age	Female Pensioners model		
	Mean Value	Prediction Error	Coefficient of variation
60	11.97	2.866	23.94%
70	9.184	3.315	36.10%
80	5.916	2.98	50.37%

We can see from table 5.4 that, for all ages, the values of the prediction error and coefficient of variation are lower than the corresponding ones when using the Sithole et al(2000) model (see Table 5.2), reflecting a lower variability in the distribution of annuity payments when the Lee-Carter mortality projection model is used. It can also be seen that the mean value of the distribution is higher for ages 60 and 70 than the corresponding ones using the Sithole et al (2000) model, while at age 80 it is slightly lower. This could be because the Lee- Carter (1992) mortality projection model, when fitted to the data in case C, produced higher mortality improvements at younger ages as compared with the Sithole et al (2000) model for the same data set.

5.4.5 Convergence diagnosis

Convergence have been checked for the three cases; A, B and C. In this section, we will show the convergence diagnosis for *Case A* only (i.e. England and Wales Male Mortality Experience, 1950-1998, inclusive, with a age grouped and classified as $\{<1, 1-4, 5-9, 10-14, \dots, 80-84, 85+\}$). For more details regarding the convergence diagnosis for cases B and C, the reader can contact the author of this thesis.

As in the pervious section, the number of updates for the simulation is such that the Monte Carlo error for each parameter is less than 5% of the sample standard deviation. Only one realization was taken of each parameter by each 10- iteration block after a burn in period of 5000 iterations. The total number of updates was 100000 iterations resulting in a sample size of $100000/10 = 10000$. Due to the structure of the prior distributions the draws of consecutive samples will be inherently correlated. So testing for autocorrelation will not help in the diagnosis of convergence. Hence, only the trace and history plots of the sample values versus iteration number and the Gelman Rubin statistic can be used to check the convergence of MCMC simulations.

Figure 5.11 gives the history plot for some selected parameters of the mortality projection model of Lee-Carter (1992) after the burn-in iterations, while figure 5.12 gives the corresponding trace plots using two sets of initial values.

Figure 5.11: History plot for some selected parameters of the Lee-Carter (1992) mortality model

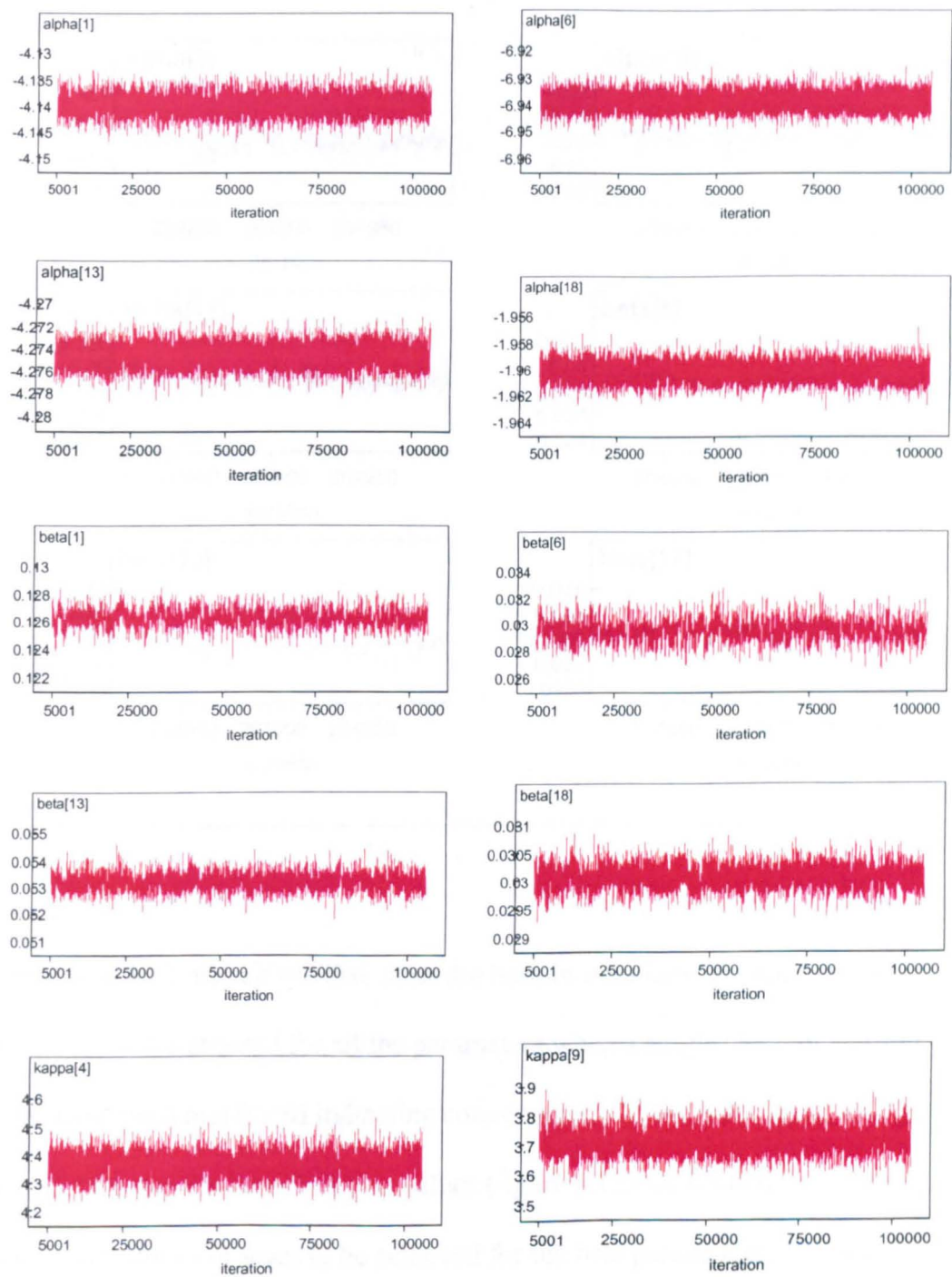
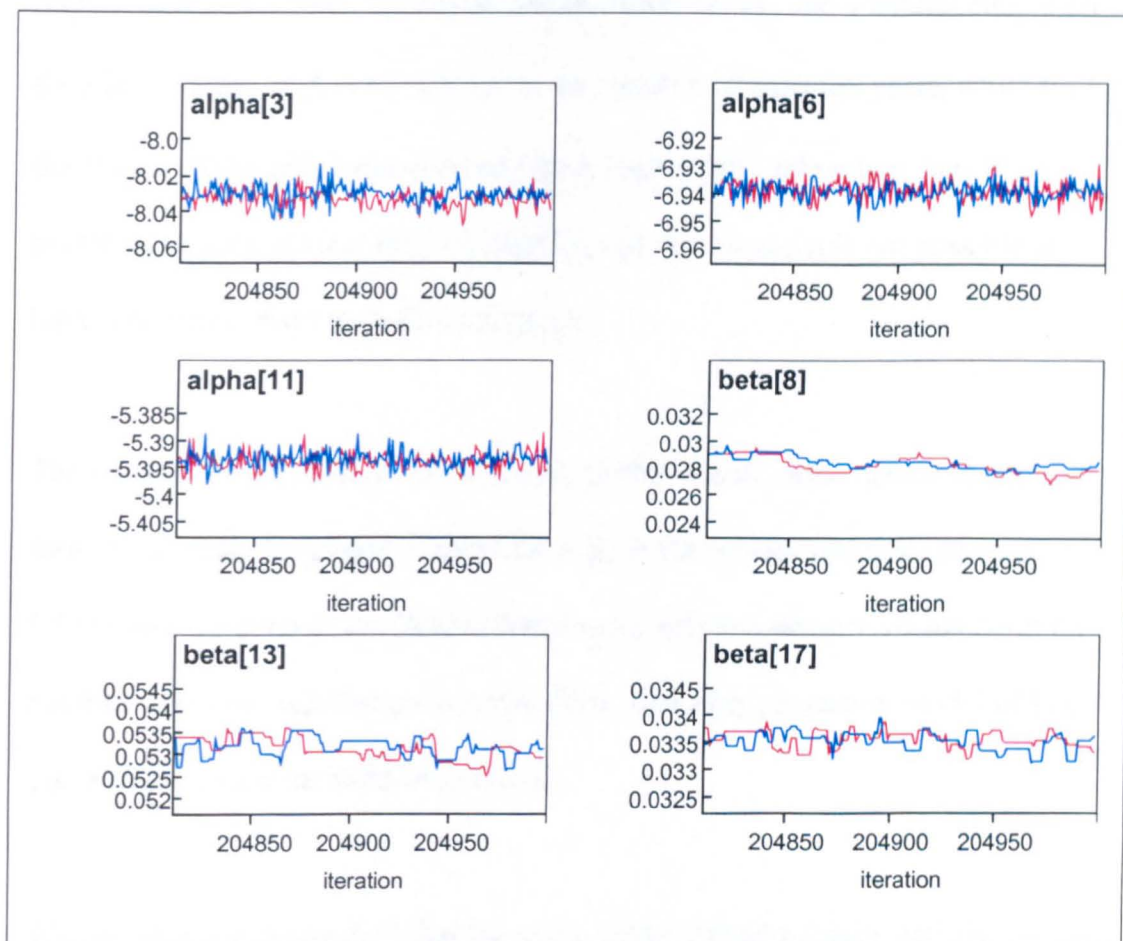


Figure 5.12: Trace plot for some selected parameters of the Lee-Carter (1992) mortality model



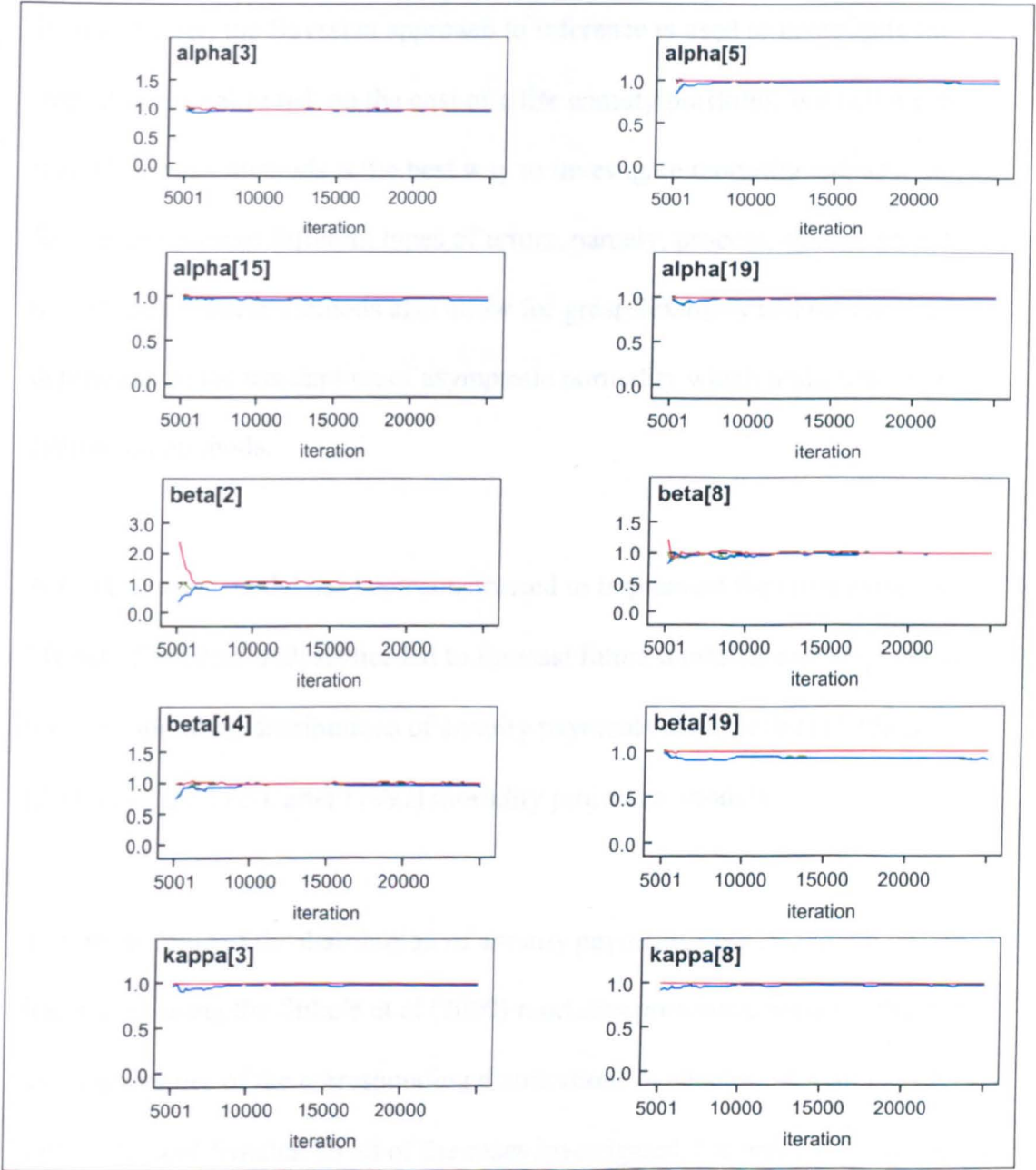
From figure 5.11 we can see that, after the burn in iterations, the simulation appears to have stabilised for all the parameters when a single chain of starting values have been monitored indicating convergence. On the other hand, using a trace plot for two chains of starting values (figure 5.12) shows that the convergence does not seem to be achieved for the beta parameters. This suggests that it may be necessary to run more iterations. Note that for Kappa the initial values have been generalized by WINBUGS itself (i.e. no initial values have

been given) this is because the average of the logarithm of the crude mortality rates at each age x could be used as initial values for α_x , and a natural choice for the initial values of β_x is $1/t$ where t is the number of calendar years, while for k , the choice of the initial values would have been totally subjective, hence we preferred to generate it using WINBUGS, and this is why it is not possible to have a multiple chain trace plot for kappa.

The Gilman-Rubin statistic can hence be used to assess convergence in a more formal way than the graphical methods used in the history and trace plots. Figure 5.13 shows the plots of the Gilman-Rubin convergence statistic versus iteration numbers for some selected parameters of the mortality projection model of Lee-Carter (1992) after the burn-in iterations.

We can see from figure 5.13 that the value of the Gelman-Rubin statistic is effectively equal to one for all of the selected parameters of the Lee-Carter (1992) mortality projection model indicating convergence of the different chains.

Figure 5.13: The Gelman-Rubin convergence statistic for some selected parameters of the Lee-Carter (1992) mortality model



5.5 Summary and remarks

In this chapter, the Bayesian approach to inference is used to investigate the impact of mortality risk on the cost of a life annuity portfolio. We believe that using Bayesian methods is the best way to investigate mortality risk as it allows for the inclusion of different types of errors, namely, process, estimation and model errors. These methods also allow for great flexibility and they are not dependent on the assumption of asymptotic normality which underlies other estimation methods.

A full Bayesian model has been constructed to implement the corresponding MCMC-Bayesian analysis needed to forecast future mortality rates and simulate the corresponding distributions of annuity payments using both the Sithole et al (2000) and the Lee-Carter (1992) mortality projection models.

The mean value of the distribution of annuity payments obtained in the Bayesian framework using the Sithole et al (2000) mortality projection model is close to the mean values of the corresponding distributions in chapter 3 for all ages for both males and females. In all of the cases investigated, the mean value obtained in the Bayesian framework was slightly higher (except for female pensioners age 60). On the other hand, the coefficient of variation of the distributions of annuity payments produced using the Sithole et al (2000) mortality projection model in

the Bayesian framework is lower than the corresponding ones in chapter 3 (except for male pensioners age 60) reflecting a lower variability in the distribution of annuity payments when the Bayesian approach to inference is used, even when the parameters are being estimated within the model (i.e. estimation error is added to the total variability).

An approach to measure parameter uncertainty (estimation error) has been allowed for by comparing the prediction error of the simulated distributions obtained from the Bayesian model when the parameters values of the Sithole et al (2000) are estimated within the model itself with the prediction error of the simulated distributions obtained when the values of the parameters are fixed at the maximum likelihood estimates. The reduction in the prediction error should then represent the estimation error- from table 5.3 we can see that the estimation error was not very significant.

Measuring model uncertainty is beyond the scope of this thesis, but as an attempt to get a taste of model uncertainty (or the effect on the distribution of annuity payments due to using a different mortality projection model) a Bayesian version of the Lee-Carter (1992) mortality projection model has been used to carry out a similar analysis as the one using the Bayesian Sithole et al (2000) mortality projection model and produce distributions of annuity payments for female life office pensioners for the period from 1983 to 1996 (Case C). The

resulting distributions of annuity payments has produced a mean value which is higher than the corresponding ones using the Bayesian version of the Sithole et al (2000) mortality projection model at ages 60 and 70, and a lower value at age 80. The coefficient of variation was lower at all ages reflecting lower variability in the distribution of annuity payments when using the Lee-Carter (1992) mortality projection model in a Bayesian framework.

Lastly, convergence has been tested as it represents an integral part of any MCMC based analysis. In this chapter, graphical methods have been used to assess convergence. These include trace and history plots, autocorrelation between consecutive draws has been also checked for the parameters of the Sithole et al (2000) mortality projection model, while it did not seem reasonable to check for autocorrelation between consecutive draws of the parameters of the Lee-Carter (1992) mortality projection model due to the structure of the prior distributions of the parameters. The Gelman-Rubin statistic has been also used as a formal test to assess convergence.

Chapter 6

Stochastic Bayesian analysis of the investment risk in a portfolio of life annuities

6.1 Introduction

We have been considering longevity risk in context of life annuities and throughout the previous chapters a fixed rate of interest has been used to measure the effect of longevity risk on the cost of a life annuity. In chapter 3, the investment risk has been investigated using sensitivity analysis based on scenarios. The aim of this chapter is to extend the analysis of the cost of life annuity to an environment in which the future rates of interest are stochastic, which will allow an integrated analysis of demographic and financial risks and their interactions.

In the actuarial literature, stochastic mortality and interest rate models for one policy only have been proposed by Panjer and Bellhouse (1980, 1981), Devolder (1986), Dufresne (1990), Norberg (1990, 1991). Generalizing the results to a portfolio is a complicated process. Some work has been done in this area, see for

example, Waters (1978), Frees (1990), Norberg (1993), and Parker (1994a, 1994b).

Selecting a stochastic model for the rate of return is not a straight forward task. It depends on the investment strategy and on how it is used. For example, if all the assets are invested in short-term fixed income securities with no default risk, then a model such as the Cox-Ingersoll-Ross (1985), which does not allow negative rates, might be acceptable. When the goal is to find the market value of some security, then term structure models would be suitable. It is also worth mentioning that when considering valuation models for annuities, or other insurance liabilities the key assumptions associated with financial models such as frictionless trading, efficient markets and the existence of a secondary market are highly suspect.

In this chapter, we will consider the effect of both longevity and investment risk on the cost of life annuities in a Bayesian framework. Future mortality improvements are modeled using the Lee-Carter (1992) projection model in a Bayesian framework as described in chapter 5. The interest rate model used is the one developed by Ballota and Haberman (2003), which makes use of a one factor Heath-Jarrow-Morton framework for the term structure of interest rate.

This chapter is organized as follows. Section 2 gives a review of the term structure of interest rates and different models for stochastic behavior of the interest rate will be discussed. Section 3 provides an analysis of annuity values in a stochastic mortality and interest rate environment in a Bayesian framework, together with a full description of the financial model that is used in the analysis is given. Section 4 provides some concluding comments.

6.2 Term structure of interest rate and Interest rate models

The modeling of the term structure of interest rates has produced a variety of approaches since the advent of arbitrage-free pricing theory and it continues to occupy the efforts of both academics and practitioners. Unlike other asset classes (for example, equities), where the lognormal Black-Scholes framework is universally accepted, no such agreement exists with regard to interest rate modeling. One reason for this is that the phenomenon we are attempting to model – the random fluctuation of the whole yield curve – is much more complex than the movements of a single stock or index price. One can intuitively relate this to the difference in the dynamics of a scalar variable (in the case of an index) and a vector (representing the yield curve). Interest rates are not a one dimensional object. In the market there are bonds with maturities between 0 and 30 years or even more. The interest received depends on the time to maturity. Generally speaking, the interest rate paid for a bond with many years to maturity

is higher than that for a bond which is close to maturity. We thus need to model interest rates using stochastic processes.

A second reason is more fundamental as far as the market in interest rate derivatives is concerned. Which consists of caps/floors and swaptions that the market prices using the Black-Scholes framework which assumes that forward Libor and swap rate are lognormal but the discount factors are non-stochastic. Thus, these instruments must be regarded as independent, where the volatility matrix for swaption prices is independent from the volatility curve associated with the cap/floor market. Also, the assumption of simultaneous lognormal behavior in the Libor and swap rates is not mathematically easy to reconcile.

6.2.1 Definitions and Notation

A zero coupon bond (sometimes referred to as a pure discount bond) is a bond with no coupon payments. Assume there is a complete set of zero-coupon bonds with maturities T in the full time interval $[0, T^*]$, given at time t a set of zero-coupon-bond prices $\{P_t(T), t < T < T^*\}$, the term structure of interest rates is the set of yields to maturity $\{r_t(T), t < T < T^*\}$, where $r_t(T)$ is defined as

$$r_t(T) = -\frac{1}{T-t} \log P_t(T) \quad (6.1)$$

for $t < T < T^*$. Clearly, $P_T(T) = 1$ for all T .

The short rate (spot rate) is the rate of interest charged on instant borrowing and lending. The short rate is a theoretical entity which does not exist in real life and can not be observed. The short interest rate r_t is defined as

$$r_t = r_t(t) = \lim_{T \rightarrow t} r_t(T) \quad (6.2)$$

The forward rate is the rate which an investor can be promised today for borrowing or lending in the future. Define the instantaneous forward rate to be:

$$f(t, T) = -\frac{\partial}{\partial T} \log P_t(T) \quad (6.3)$$

The function $f(t, T)$ corresponds to the rate we can contract for at time t on a riskless loan that begins at time T and is returned an instant later. The short rate r_t is contained in the forward rate structure since $r_t = f(t, t)$. Note that zero coupon bond prices and forward rates represent equivalent information as we can see in equation (6.4).

$$P_t(T) = \exp \left(- \int_t^T f(t, s) ds \right) \quad (6.4)$$

6.2.2 Interest rate models

Interest rates and their dynamics provide probably the most computationally difficult part of modern financial theory. When choosing a model, one could pose some questions such as:

- Does the dynamics imply positive rates for the short rate r_t ?
- Is the model mean reverting? (i.e. fluctuating around a long term mean)
- What does the volatility structure implied by the model look like?
- How suited is the model for Monte Carlo simulations?

These points are essential for the understanding of the theoretical and practical implications of interest rate models and hence specifying dynamics, volatilities and whether a one-factor or a multi-factor model should be used as well as the choice of the bond pricing framework to be used, which is determined partly by the actual variable used to describe the model, and can be categorized into three families: spot rate, forward rate and market models.

Although all three of these prescriptions are mathematically consistent (by definition of a term structure model), each approach leads to distinct development, implementation and calibration issues.

In this chapter, we will restrict the discussion to default-free term structures, and interest rates driven purely by Brownian motion (i.e. not including jump processes). We can require that the interest rate market (or equivalently the bond market) from an economic point of view to follow the principles below:

- The zero-coupon bond price is strictly positive;
- No default is possible so $P(T,T) = 1$;
- There is no arbitrage in the market.

At the same time we require the model for interest rate to:

- Capture a wide range of realistic future term structures;
- Be numerically efficient and practical to use within risk management systems.

Spot rate models (pioneered by Vasicek (1977)) attempt to describe the bond dynamics through directly modeling the short-term interest rate. Heath Jarrow and Morton (HJM) (1990, 1992) established a general framework where the above principles are satisfied. This framework directly uses the arbitrage free dynamics of the entire zero bond or equivalently, the term structure of forward rates to price interest rate derivatives. Market models are a class of models within the HJM framework that describe variables directly observed in the

market, such as the discretely compounding Libor and swap rates. These approaches have historically developed according to this order. It can be seen that the models have evolved along this path of increasing sophistication (from spot models initially to market models). In this section, we will briefly discuss the similarities and differences in terms of formulation of each model category (spot rate, forward rate and market models).

In all of the model categories, the yield curve is described through stochastic differential equations driven by a diffusion term and a drift term. Based on the arbitrage-free principle, the market price of risk is removed by the choice of the drift. This is performed in different ways. Spot rate models match the initial yield curve that implicitly holds information on investor choice and hence the market price of risk, through the drift function. Models formulated with instantaneous forward rates explicitly relate the choice of volatility function to the form of the drift (imposed through the HJM condition), in order for the no-arbitrage principle to hold. Similarly, for market models the drift is adjusted to ensure that the model remains arbitrage-free.

6.2.2.1 Short rate (spot rate) models

Historically it has been the short interest rate that has been modeled as the basic process. The first generation of models developed was generally spot rate-based. This choice was due to a combination of mathematical convenience and tractability, or numerical ease of implementation. Furthermore, the most widely used of these models are the one-factor models, in which the entire yield curve is specified by a single stochastic state variable, in this case the spot or short-term rate. An example of these includes the model due to Vasicek (1977), whereby the short rate is modeled as a normal mean reverting process with constant parameters. Vasicek (1977) assumed that the instantaneous spot rate under the risk neutral measure follows an Ornstein-Uhlenbeck process with constant coefficients, hence:

$$dr(t) = k[\theta - r(t)]dt + \sigma dW(t), \quad r(0) = r_0 \quad (6.5)$$

where r_0 , k , θ and σ are positive constants and $W(t)$ is a standard Brownian motion. Integrating equation (6.5), we obtain for each $s \leq t$,

$$r(t) = r(s)e^{-k(t-s)} + \theta(1 - e^{-k(t-s)}) + \sigma \int_s^t e^{-k(t-u)} dW(u) \quad (6.6)$$

Thus, $r(t)$ conditional on a filtration F_s ($s \leq t$) normally distributed with mean and variance given respectively by

$$\begin{aligned}
E\{r(t)|F_s\} &= r(s)e^{-k(t-s)} + \theta(1 - e^{-k(t-s)}) \\
Var\{r(t)|F_s\} &= \frac{\sigma^2}{2k} [1 - e^{-2k(t-s)}]
\end{aligned} \tag{6.7}$$

This implies that, for each time t , the short rate $r(t)$ can be negative with a positive probability, which represent a major drawback of the Vasicek model. However, the analytical tractability that is implied by a Gaussian density is not achieved when assuming other distribution for the process $r(t)$. As a consequence of (6.7), the short rate $r(t)$ is mean reverting, since the expected rate tends, for t going to infinity, to the value θ .

The general equilibrium approach developed by Cox-Ingersoll-Ross (CIR) (1985) led to the addition of a square-root diffusion term to the Vasicek (1977) model. The resulting model has been a benchmark for many years because of its analytical tractability and the fact that, contrary to the Vasicek(1977) model the short rate is always positive. The model formulation under the risk neutral measure is

$$dr(t) = k[\theta - r(t)]dt + \sigma\sqrt{r(t)}dW(t), \quad r(0) = r_0 \tag{6.8}$$

where r_0, k, θ and σ are positive constants, the condition $2k\theta > \sigma^2$ has to be imposed to ensure that the short rate remains positive.

Ho & Lee (1986) pioneered an arbitrage-free lattice approach for interest rate models taking initial term structure of interest rate as an input. Hull & White

(1990) extended the Vasicek model to fit both the current term structure and volatilities of interest rates. In their model the short rate follows a normal mean reverting process with time dependent parameters. The Hull & White (1990) model is popular in practice as it produces closed form solution for bond prices. Black, Derman & Toy (BDT)(1990), combined the mean reverting property of the short rate with a lognormal distribution. The major appeal of the model is the transparent calibration procedure to the yield and volatility curves. However, the cost of that was mutually dependent mean reversion and volatility terms. Black-Karasinski (1991) rectified this shortcoming of the BDT model by allowing for independent parameters. For a detailed description of the different models proposed, together with their extensions, the reader can refer to Björk (1998).

All of the above models are driven by one-dimensional stochastic processes. Moreover, the volatility structure in them is always deterministic and stationary. In reality, the log returns of zero coupon bonds of different maturities are not perfectly correlated. Multi-factor models can take this into account. The use of multiple factor and path dependent volatility functions gives a certain flexibility since it can incorporate changes in the level and the slope of the term structure, though with each extra factor there is a considerable increase in complexity. In practice, we have to trade off precision and numerical tractability.

6.2.2.2 Forward rate models

An alternative approach to modeling the term structure was offered by the Heath, Jarrow and Morton (HJM) (1992) structure. In contrast to the short rate approach, they model the entire yield curve as a state variable, providing conditions in a general framework that incorporates all the principles of arbitrage-free pricing and discount bond dynamics. Moreover, the HJM model uses all the information available in the initial term structure. The HJM methodology uses as the driving stochastic variable the instantaneous forward rates, the evolution of which is dependent on a specific (usually deterministic) volatility function.

Because of the relationship between the spot rate and the forward rate, any spot rate model is also an HJM model. In fact, any interest rate model that satisfies the principles of arbitrage-free bond dynamics must be within the HJM framework.

Heath, Jarrow and Morton (1992) assumed that, for a fixed maturity T , the instantaneous forward rate $f(t, T)$ evolves under a given measure according to the following diffusion process:

$$\begin{aligned} df(t, T) &= \alpha(t, T)dt + \sigma(t, T)dW(t), \\ f(0, T) &= f^M(0, T) \end{aligned} \tag{6.9}$$

where $f^M(0, T)$ is the market instantaneous forward rate at time $t = 0$ for maturity T , and where $W = (W_1, \dots, W_N)$ is an N -dimensional Brownian motion allowing the model to consider various factors. $\alpha(t, T)$ is the drift function and $\sigma(t, T)$ is the volatility function.

The advantage of modeling forward rate as in (6.9) is that the current term structure of rates is by construction an input of the selected model.

Heath, Jarrow and Morton proved that, in order for a unique equivalent martingale measure to exist, the function α can not be arbitrarily chosen, but it must be equal to a quantity depending on the vector volatility σ and on the drift rates in the dynamics of N selected zero coupon bond prices. In particular, if the dynamics in (6.9) are under the risk neutral measure, then we must have

$$\alpha(t, T) = \sigma(t, T) \int_t^T \sigma(t, s) ds = \sum_{i=1}^N \sigma_i(t, T) \int_t^T \sigma_i(t, s) ds \quad (6.10)$$

So that the integrated dynamics of $f(t, T)$ under the risk neutral measure are fully specified once the vector volatility function σ is provided, such that;

$$\begin{aligned} f(t, T) &= f(0, T) + \int_0^t \sigma(u, T) \int_u^T \sigma(u, s) ds du + \int_0^t \sigma(s, T) dW(s) \\ &= f(0, T) + \sum_{i=1}^N \int_0^t \sigma_i(u, T) \int_u^T \sigma_i(u, s) ds du + \sum_{i=1}^N \int_0^t \sigma_i(s, T) dW_i(s) \end{aligned} \quad (6.11)$$

Given these dynamics of the instantaneous forward rate $f(t, T)$, application of Ito's lemma gives the following dynamics of the zero coupon bond price $P_t(T)$:

$$dP_t(T) = P_t(T) \left[r(t)dt - \left(\int_0^T \sigma(t, s)ds \right) dW(t) \right] \quad (6.12)$$

where $r(t)$ is the instantaneous short term interest rate at time t , that is

$$\begin{aligned} r(t) = f(t, t) &= f(0, t) + \int_0^t \sigma(u, t) \int_u^t \sigma(u, s) ds du + \int_0^t \sigma(s, t) dW(s) \\ &= f(0, t) + \sum_{i=1}^N \int_0^t \sigma_i(u, t) \int_u^t \sigma_i(u, s) ds du + \sum_{i=1}^N \int_0^t \sigma_i(s, t) dW_i(s) \end{aligned} \quad (6.13)$$

6.2.2.3 Market models

Market models are considered one of the most popular families of interest rate models due to the agreement between these models and market formulas for two basic derivative products (caps and swaptions). Hence, market models are more appealing in this context than HJM framework which is based on continuously compounded rates and is therefore fundamentally different from actual forward Libor and swap rates as traded in the market. The lognormal HJM model is also well known to exhibit unbounded behavior (producing infinite values) in contrast to the use of lognormal Libor distribution in Black's formula for caplets. The construction of a mathematically consistent theory of a term structure with

discrete Libor rates being lognormal was achieved by Sandmann & Sondermann (1993). Brace, Gatarek & Musiela (1997) and Jamshidian (1997) (BGM/J) developed a unifying framework, the market model, based on HJM, for forward LIBOR rates assuming simple compounding of LIBOR rates instead of continuous compounding of forward rates under the HJM which would result in an exploding process. This approach is arbitrage free and has a closed form solutions for European swaptions.

6.3 Bayesian analysis of annuity values in a stochastic interest rate environment

In this section, a fully Bayesian model is used analyze annuity values in the same manner as in chapter 5. The difference is that, instead of considering longevity risk only, the effect of investment risk will be allowed for through the modeling of the interest rate used in discounting as a stochastic process. The Lee-Carter mortality projection methodology, using the Poisson log-bilinear methodology (described in chapter 5), will be used in a Bayesian framework to forecast future forces of mortality, and hence produce full distribution of annuity values. The likelihood function and the prior assumptions for the parameters of the mortality model are as in chapter 5. The analysis has been performed using the CMI data for female life office pensioners, 1983-1996 inclusive, with age classified by individual year, 60-95 inclusive, so that the results can be compared to the corresponding ones (case C) in chapter 5.

We will start by describing the financial model used together with the prior distribution of its parameters.

6.3.1 The financial model

The financial model used is the same as the model used by Ballotta and Haberman (2003) to evaluate guaranteed annuity options.

Under the condition of a frictionless market with continues trading, no taxes, no transaction costs, no restrictions on borrowing or short sales and perfectly divisible securities. Assume that the evolution of the forward rate is modelled in a single-factor Heath-Jarrow-Morton (HJM) framework (Heath et al (1992)), i.e.

$$df(t, T) = \left(\sigma_f(t, T) \int_t^T \sigma_f(t, u) du \right) dt + \sigma_f(t, T) dW_t \quad (6.14)$$

where the volatility function $\sigma_f(t, T)$ is a F_t adapted satisfying

$\int_0^T \sigma_f^2(s, T) ds < \infty$ and $(W_t : t \geq 0)$ is a standard one dimensional Brownian

motion. Under these assumptions, the price of a zero coupon bond with

redemption at time T is $P_t(T) = \exp\left(-\int_t^T f(t, s) ds\right)$ and the money market

account is given by $B_t = \exp\left(\int_0^t r_s ds\right)$ where r_t is the short rate.

We assume that the volatility of the forward rate follows an exponentially decaying structure, such that

$$\sigma_f(t, T) = \sigma e^{-\lambda(T-t)} \quad (6.15)$$

where $\sigma > 0$, $\lambda > 0$. Hence, the forward rate dynamics is given by

$$df(t, T) = \left(\sigma^2 e^{-\lambda(T-t)} \int_t^T e^{-\lambda(u-t)} du \right) dt + \sigma e^{-\lambda(T-t)} dW_t \quad (6.16)$$

Under these assumptions it follows that the short rate process can be expressed as:

$$r_t = f(0, t) + (1 - e^{-\lambda t}) \left[\frac{\sigma^2}{2\lambda^2} (1 - e^{-\lambda t}) \right] + \sigma \int_0^t e^{-\lambda(t-v)} dW_v \quad (6.17)$$

Equation (6.17) shows that the exponentially decaying structure of the forward rate volatility leads to a mean reverting form of the short rate that closely resemble an extended version of the Vasicek(1977) model.

Notice that since $v \in [0, t]$, then:

$$\begin{aligned} \int_0^t e^{-\lambda(t-v)} dv &= \int_t^T e^{-\lambda(u-t)-\lambda(t-v)} du = e^{-\lambda(t-v)} \int_t^T e^{-\lambda(u-t)} du \\ &= e^{-\lambda(t-v)} \left(\frac{1 - e^{-\lambda(T-t)}}{\lambda} \right) = e^{-\lambda(t-v)} \gamma(t, T) \end{aligned} \quad (6.18)$$

where $\gamma(t, T) = \left(\frac{1 - e^{-\lambda(T-t)}}{\lambda} \right)$.

Using equation (6.17) and following, for example, Jarrow and Turnbull (1994) and Chiarella and Kwon (2001), we can establish that

$$(r_t - f(0, t)) \sim N(m_r(t), \sigma_r(t))$$

where

$$m_r(t) = (1 - e^{-\lambda t}) \left[\frac{\sigma^2}{2\lambda^2} (1 - e^{-\lambda t}) \right], \quad \sigma_r^2(t) = \sigma^2 \left(\frac{1 - e^{-2\lambda t}}{2\lambda} \right).$$

Therefore

$$P_t(T) = \frac{P_0(T)}{P_0(t)} e^{-(1/2)r^2(t,T)\sigma_r^2(t) - r(t,T)(r_t - f(0,t))} \quad (6.19)$$

Although Equations (6.17) and (6.19) are similar to the expressions derived by Vasicek (1977), they differ in the fact they are obtained taking the initial term structure as exogenous, while for the Vasicek model the initial term structure is endogenous.

Full details regarding the model and its detailed derivation can be found in Ballotta and Haberman (2003).

6.3.2 Methodology and prior distributions

For the mortality model, we will use the same likelihood function and prior assumptions of the Lee-Carter (1992) mortality projection model as in chapter 5. As for the investment model, equation (6.19) will be used to find at time t the set of zero-coupon-bond prices $\{P_t(T), t < T\}$, which can then be used as the discount factors for different durations T in the future. We are going to use the set of zero coupon bond prices at time $t=1$, i.e. we need to find the values for $P_1(T)$, where $\{1 < T\}$. By substituting $t=1$ in equation (6.19), $P_1(T)$ can be expressed as;

$$P_1(T) = \frac{P_0(T)}{P_0(1)} e^{-(1/2)\gamma^2(1,T)\sigma_r^2(1) - \gamma(1,T)(r_1 - f(0,1))} \quad (6.20)$$

where

$$\gamma(1,T) = \left(\frac{1 - e^{-\lambda(T-1)}}{\lambda} \right)$$

$$(r_1 - f(0,1)) \sim N(m_r(1), \sigma_r(1))$$

$$m_r(1) = (1 - e^{-\lambda}) \left[\frac{\sigma^2}{2\lambda^2} (1 - e^{-\lambda}) \right], \quad \sigma_r^2(1) = \sigma^2 \left(\frac{1 - e^{-2\lambda}}{2\lambda} \right).$$

As in Ballotta and Haberman (2003), in order to compute the initial bond prices $P_0(T)$ a flat initial term structure is assumed and fixed at 4%, i.e.

$f(0, \cdot) = f_0 = 0.04$. The prior distribution chosen for λ is the uniform distribution over the range (0.1, 0.15). The Gamma distribution naturally was used as the prior distribution for σ , with a mean of 0.01 and a small variance (in line with what have been assumed in Ballotta and Haberman (2003)). Note that these assumptions were made so that the model resembles the current market conditions in the UK.

6.3.3 Numerical results

In line with the calculations performed in chapter 5, we will consider the CMI data which relate to female office pensioners for the period from 1983 to 1996. We will consider a single life annuity at ages 60, 70 and 80 with payments of £1 due at the end of each year.

The estimated values for the parameters of the mortality projection model α_x, β_x and κ_t are obviously the same as results for case C in chapter 5 as the same set of data, likelihood function and prior assumptions are being used.

As for the interest rate model, figure 6.1 shows the zero coupon bond prices $P_1(T)$ at time $t=1$ for different future durations T that have been calculated using equation 6.20.

Figure 6.1: Estimated values for zero coupon bond prices at time $t=1$ for different maturities T

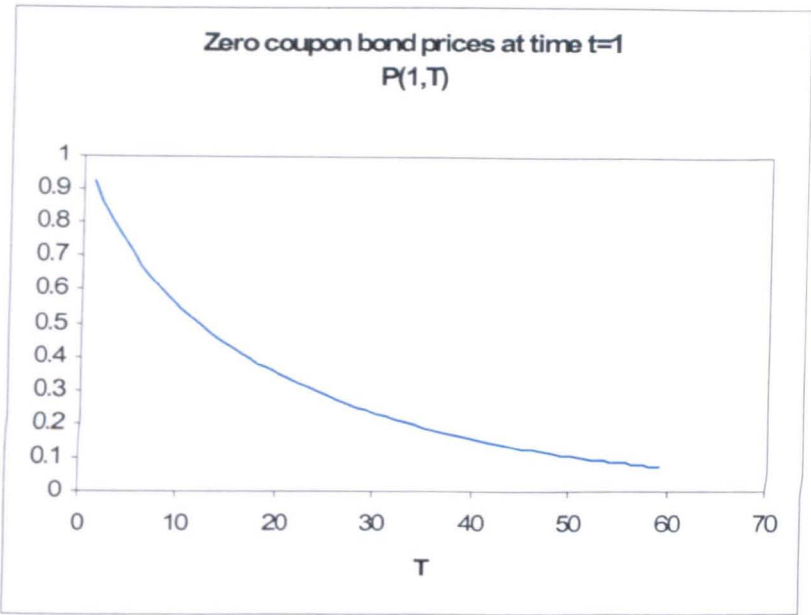
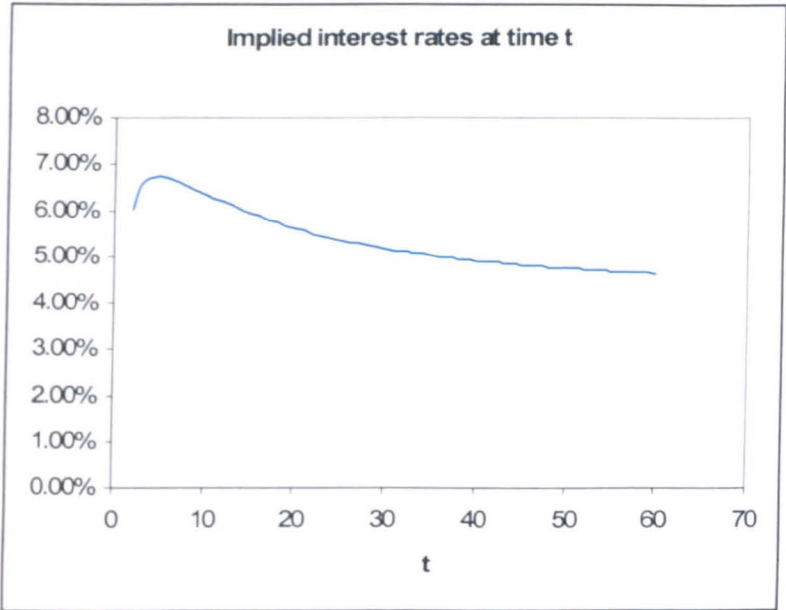


Figure 6.1 shows that as T increases the value of $P_1(T)$ decreases. The values of zero coupon bonds at time 1 for different durations T represent the discount factor at time $T-1$, i.e. $P_1(T) = \left(\frac{1}{1+i}\right)^{(T-1)}$. Note that the financial model described 6.3.1 allows negative rates of interest, which could result in a discount factor which is higher than 1. A constraint was imposed to prevent this from happening and as shown in figure 6.1 the Zero coupon bond prices are behaving as they suppose to. Figure 6.2 shows the interest rates implied by the corresponding zero coupon bond prices.

Figure 6.2: Interest rates implied by zero coupon bond prices at different maturity time T



It can be seen from figure 6.2 that the rate of interest is higher than 6% for early durations and then it decreases, approaching the long term mean of 4%.

A summary of the descriptive statistics of the simulated distributions of annuity payments for female pensioners using a Bayesian version of the Lee-Carter (1992) mortality projection model in a stochastic interest rate environment is shown below in table 6.1.

Table 6.1: Summary of the descriptive statistics of the simulated distributions of annuity payments for female life office pensioners

Age	Female Pensioners model		
	Mean Value	Prediction Error	Coefficient of variation
60	12.11	3.104	25.63%
70	9.061	3.345	36.92%
80	5.836	2.917	49.98%

We can see from table 6.1 that the mean value of the distribution of annuity payments at age 60 is higher than the mean value of the corresponding distribution which assumed a fixed rate of interest of 6% for discounting the cash flows (table 5.4). The reason is that, although at early durations the rate of interest used in discounting is higher than 6%, resulting in a lower value for the cash flow for these durations as compared to the case when a fixed rate of 6% is being used. This effect has been offset by using a lower interest rate, and hence a higher value of the cash flows at later durations (see figure 6.2). This results in an overall increase in the mean value of the distribution at age 60. On the other hand, the mean value of the distribution of annuity payments at ages 70 and 80 is lower than the corresponding ones when the rate of interest was fixed at 6% because the rate of interest used was on average higher than 6% in most of the years. It can also be seen from table 6.1 that the coefficient of variation at ages 60 and 70 is higher than the corresponding ones in chapter 5 reflecting a higher variability in the distribution of annuity payments when the rate of interest is treated as a random quantity. At age 80, the coefficient of variation is lower than

the one in chapter 5 as the additional variability attributed due to the randomness of the interest rate used in compounding is of a lesser importance than both ages 60 and 70 because the interest rate is on average much higher than 6%, and the higher the rate of interest the lower the variability of the distribution of annuity payments.

6.3.4 Convergence diagnostics

In this section, we will consider the convergence of the parameters of the financial model only. As before, the number of updates that the simulation has to be run for is such that the Monte Carlo error for each parameter is less than 5% of the sample standard deviation. In this chapter the number of burn-in iterations was 5000, and the total number of updates afterwards was 100,000 and to reduce any effect of autocorrelation between consecutive samples we retain only one realization in every 10 iterations for each parameter. As in chapter 5, trace and history plots of the sample values versus iteration and the Gelman Rubin statistic were used to check the convergence of the MCMC simulations of the financial model. The autocorrelations between the draws of consecutive samples of all the parameters involved was also checked for.

Figure 6.3 gives the history plot for some parameters of the financial model and some of the zero coupon bond prices at time $t=1$ for different durations T in the

future ($P_1(T)$) at selected durations after the burn in iterations, while figure 6.4 gives the corresponding trace plots using two sets of initial values.

Figure 6.3: History plot for the parameters of the financial model and zero coupon bond prices ($P(1,T)$) at time $t=1$ for some selected maturities T

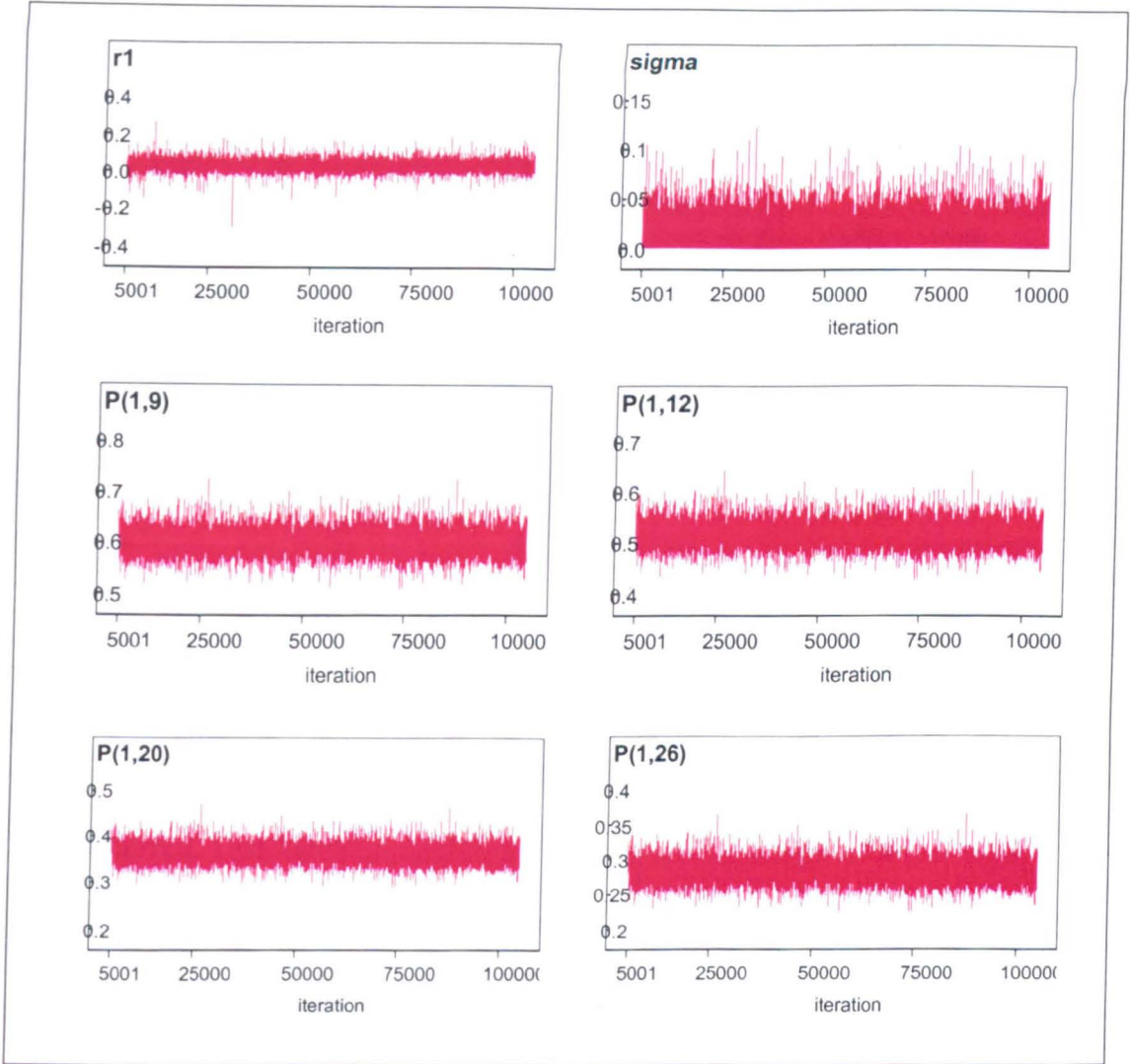
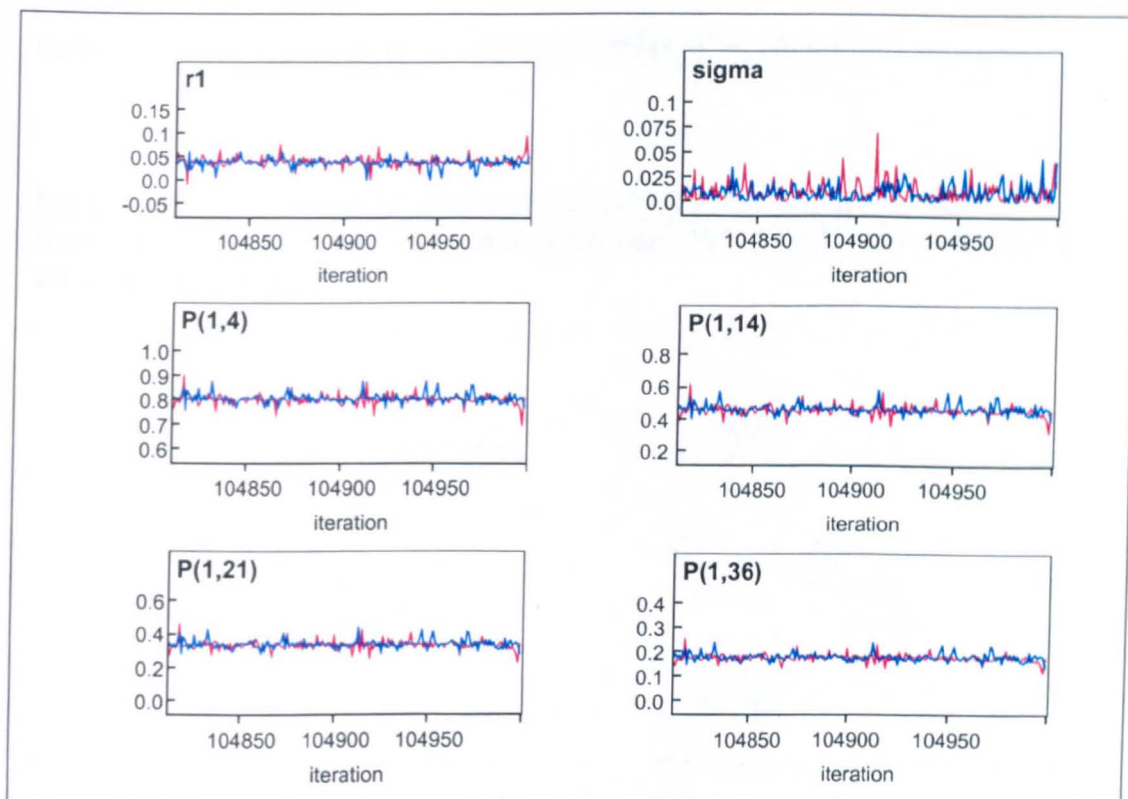


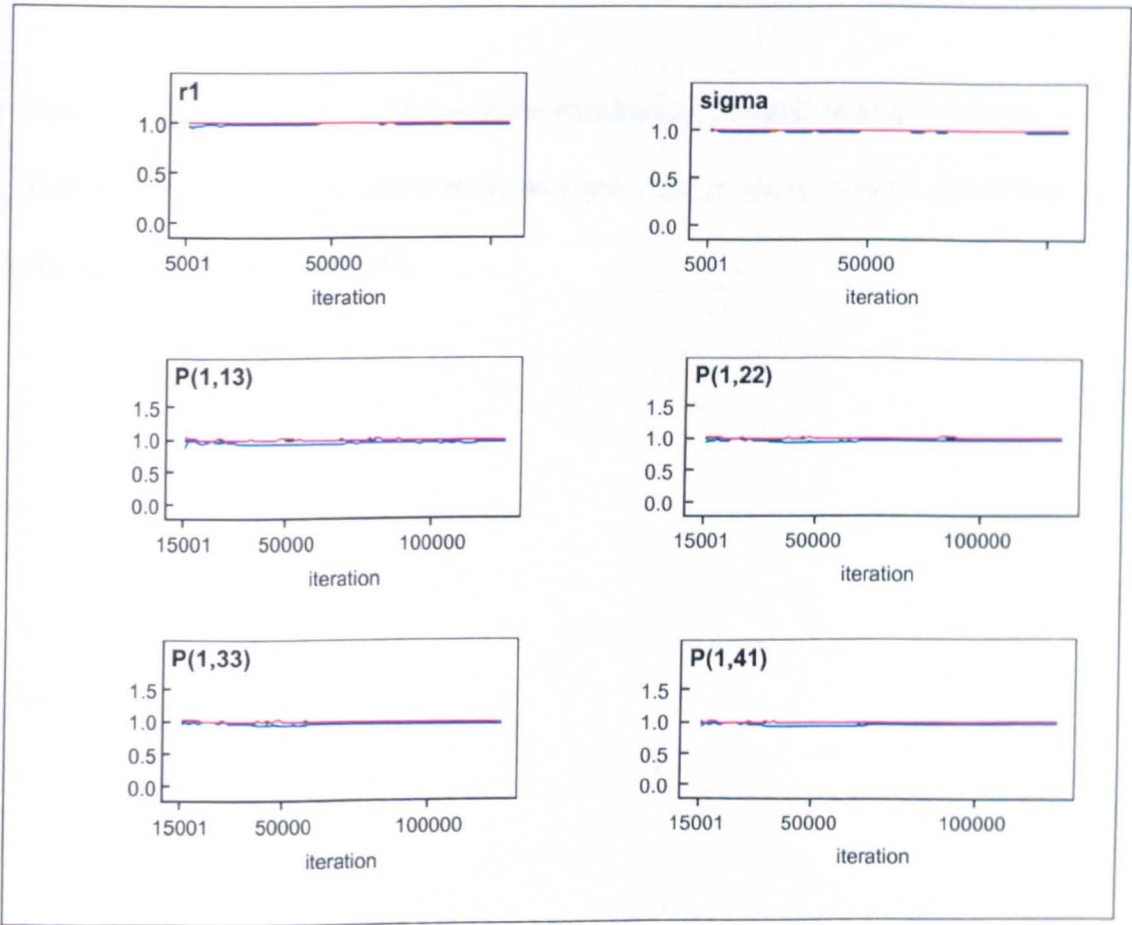
Figure 6.4: Trace plot for the parameters of the financial model and zero coupon bond prices ($P(1, T)$) at time $t=1$ for some selected maturities T



From figure 6.3 we can see that, after the burn-in iterations, the simulation appears to have stabilised for all observed variables indicating that convergence has occurred. This has been confirmed by using two chains of starting values (figure 6.4) where the simulations appear to exhibit the same behavior through the iterations irrespective of the initial values used.

Figure 6.5 shows the plots of the Gelman-Rubin convergence statistic versus iterations number for the parameters of the financial model and zero coupon bond prices ($P(1,T)$) at time $t=1$ for some selected maturities T .

Figure 6.5: Gelman-Rubin convergence statistic for the parameters of the financial model and zero coupon bond prices ($P(1,T)$) at time $t=1$ for some selected maturities T

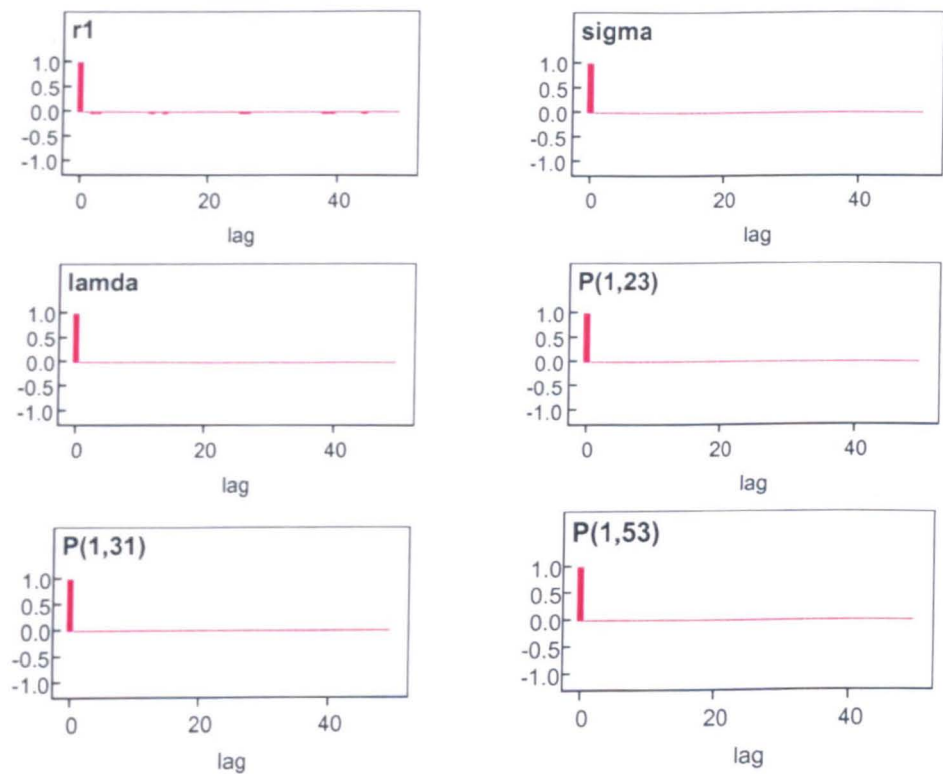


We can see from figure 6.5 that the value of the Gelman-Rubin statistic is effectively equal to one, indicating convergence of the different chains.

The autocorrelations between the draws of consecutive samples need to be checked as well to ensure that the Gibbs Sampler will not be slow to explore the entire posterior distribution. As we mentioned before, the autocorrelation function measures how correlated the values in the chain are with their close neighbours. An independent chain will have approximately zero autocorrelation at each lag.

Autocorrelations up to lag 50 for some parameters of the financial model at selected durations show that there is no significant autocorrelations remaining in the samples. (See figure 6.6).

Figure 6.6: Autocorrelation function for the parameters of the financial model and zero coupon bond prices ($P(l, T)$) at time $t=l$ for some selected maturities T



6.4 Summary and remarks

In this chapter, the analysis of the change of the cost of life annuity has been extended to an environment in which the future rates of interest are stochastic which allows an integrated analysis of both longevity and investment risks and their interaction. This analysis is performed in a Bayesian framework, with future mortality improvements being modeled using the Lee-Carter (1992) projection as described in chapter 5. The interest rate model used is the one

developed by Ballota and Haberman (2003), which makes use of a one factor Heath-Jarrow-Morton framework for the term structure of interest rate. The data used are the CMI data for female life office pensioners, 1983-1996 inclusive, with age classified by individual year, 60-95 inclusive, so that the results can be compared to the corresponding ones of case C in chapter 5.

The resulting rates of interest are higher than 6% for early durations and then decreases, approaching the long term mean of 4%. This leads to a higher mean value for the distribution of annuity payments at age 60 as compared to the corresponding one in chapter 5 (case C), while at ages 70 and 80 the mean values of the distribution of annuity payments are lower than the corresponding ones in chapter 5 (case C). This is because the remaining life time is longer at age 60 as compared to ages 70 and 80, so that the higher interest rates that have been used to discount cash flows at early durations are offset by lower rates at the later durations. At ages 70 and 80, the remaining life time is on average not long enough for this to have an effect.

The coefficient of variation of the distributions of annuity payments is higher at ages 60 and 70 than the corresponding ones in chapter 5 (case C), reflecting a higher variability in the distribution of annuity payments that could be attributed to treating the interest rate used for discounting at different duration as a random variable instead of fixing it to 6% as before. At age 80 the coefficient of variation was slightly lower than the corresponding one in chapter 5 (case C), as the effect

of the additional variability due to modeling the rate of interest as a stochastic quantity is of lesser importance especially when the rate of interest used was on average higher than 6%.

Chapter 7

Conclusions

7.1 Summary

This chapter is organised as follows: the main findings and the conclusions in each of the previous chapters are summarised in this section. Section 7.2 presents a discussion of possible extensions and further work.

Chapter 3 – Analysing the distribution of life annuities using simulation techniques. In this chapter simulation techniques have been used to obtain a distribution of annuity payments by modelling a particular path that a group of persons may follow during their life time by allowing the time of death for each person to be a random variable. The effect of age at inception, gender, assumed interest rate and the level of mortality have been investigated using scenario analysis. The properties of the distributions of outcomes of each of the scenarios have been analysed using UK life office pensioners' data from the Continuous Mortality Investigation Bureau and mortality improvements have been allowed for using the Sithole et al (2000) mortality projection model.

Using the simulated distributions, we can draw the following conclusions:

- The distribution of the present value of annuity payments for female pensioners is less variable than in the corresponding one for male pensioners at same age.
- As age at inception increases, the coefficient of variation increases and the effect of mortality improvement decreases and the shapes of the distributions become similar to each other.
- For younger ages at inception, the shape of the distribution of present value of annuity payments is negatively skewed. With increasing age, the distribution becomes less negatively skewed until, at some point, it becomes positively skewed.
- The effect on the present value of annuity payments of living longer than average is increased by the lower discounting applied to the future payments in a low interest environment.
- Decreasing the reduction factors (i.e. assuming higher mortality improvements.) has a stronger effect on the additional cost of annuity than increasing the reduction factors by the same percentage.

Chapter 4 – Entropy, Longevity and the cost of life annuity. The aim of this chapter is to develop a theory that extends the use of the entropy measure applied in population biology by Demetrius (1976) to measure the effect of a proportionate changes in the force of mortality on the cost of life annuity at different levels of interest rate and mortality improvements, this represent the

first main contribution of this chapter. Using this new theory allows different sources of risk in a life annuity contract to be summarized in a one figure index (i.e. the entropy measure H). The approach of Keyfitz (1977) has been extended to obtain numerical values for H . This has been applied to the English Life Tables over the period from 1851 to 1991 and also by applying different mathematical models for mortality projections such as the Gompertz and the Sithole et al (2000) mortality projection models for both males and females aged 60.

The numerical results for H suggests that, at very high or low levels of mortality, the effect of mortality changes on the value of life annuity is of less importance. This reflects the fact that when mortality is already very high or very low, any change (whether an increase or a decrease in the force of mortality) is less likely to have a significant effect on the cost of life annuity. This means that – theoretically - even if mortality continues to improve it will reach a level beyond which any more improvements would not affect the cost of survival benefits by much.

Chapter 5 – Bayesian analysis of the changes in the cost of life annuity due to longevity risk. In Chapter 5, the effect of longevity risk has been investigated in the context of life annuities in a Bayesian framework. The simulated distributions of annuity payments are obtained in a manner similar to that in

chapter 3 except that the Bayesian approach combines the estimation of the parameters of the mortality projection models together with the simulation of the annuity cost, so that parameter uncertainty will be allowed for. The analysis has been performed using both the Sithole et al (2000) and the Lee-Carter (1992) mortality projection models. An approach to measure the effect of parameter uncertainty has been introduced in this chapter.

Numerical results showed that, the mean value of the distribution of annuity payments obtained in the Bayesian framework using the Sithole et al (2000) mortality projection model is close to the mean values of the corresponding distributions in chapter 3 for all ages for both males and females. And that, In all of the cases investigated, the mean value obtained in the Bayesian framework was slightly higher (except for female pensioners age 60). On the other hand, the coefficient of variation of the distributions of annuity payments produced using the Sithole et al (2000) mortality projection model in the Bayesian framework is lower than the corresponding ones in chapter 3 (except for male pensioners age 60) reflecting a lower variability in the distribution of annuity payments when the Bayesian approach to inference is used, even when the parameters are being estimated within the model (i.e. estimation error is added to the total variability).

Parameter uncertainty (estimation error) has been measured for the Sithole et al (2000) mortality projection model, and numerical results suggested that estimation error was not very significant (see table 5.3).

An approach to measure parameter uncertainty (estimation error) has been allowed for by comparing the prediction error of the simulated distributions obtained from the Bayesian model when the parameters values of the Sithole et al (2000) are estimated within the model itself with the prediction error of the simulated distributions obtained when the values of the parameters are fixed at the maximum likelihood estimates. The reduction in the prediction error should then represent the estimation error- from table 5.3 we can see that the estimation error was not very significant.

A Bayesian version of the Lee-Carter (1992) mortality projection model has been used to carry out a similar analysis as the one using the Bayesian Sithole et al (2000) mortality projection model and produce distributions of annuity payments. The resulting distributions of annuity payments has produced a mean value which is higher than the corresponding ones using the Bayesian version of the Sithole et al (2000) mortality projection model at ages 60 and 70, and a lower value at age 80. The coefficient of variation was lower at all ages reflecting lower variability in the distribution of annuity payments when using the Lee-Carter (1992) mortality projection model in a Bayesian framework.

Chapter 6 – Stochastic Bayesian analysis of the investment risk in a portfolio of life annuity. In chapter 6 a model for the rate of interest assumed has been incorporated to reflect the effect of the uncertainty of future returns on the cost of a life annuity. As in chapter 5, this analysis is performed in a Bayesian frame work in which the parameters of both the financial model and the mortality projection models are estimated. Mortality improvements are allowed for using the Lee-Carter (1992) mortality projection model and the interest rate model used is the one developed by Ballotta and Haberman (2003).

In chapter 5 a rate of interest of 6% has been used for discounting annuity payments for all durations. On the other hand, the resulting rates of interest using the stochastic model are higher than 6% for early durations and then started to decrease approaching the long term mean of 4%. This leads to higher mean value of the distribution of annuity payments at age 60 as compared to the corresponding one in chapter 5 (case C), while at ages 70 and 80 the mean values of the distribution of annuity payments are lower than the corresponding ones in chapter 5 (case C). This is because the remaining life time is longer at age 60 as compared to ages 70 and 80, so that the higher interest rates that have been used to discount cash flows at early durations are offset by lower rates at the later durations. At ages 70 and 80, the remaining life time is on average not long enough for this to have an effect.

The coefficient of variation of the distributions of annuity payments is higher at ages 60 and 70 than the corresponding ones in chapter 5 (case C), reflecting a higher variability in the distribution of annuity payments that could be attributed to treating the interest rate used for discounting at different duration as a random variable instead of fixing it to 6% as before. While at age 80 the coefficient of variation was slightly lower than the corresponding one in chapter 5 (case C), as the effect of the additional variability due to modeling the rate of interest as a stochastic quantity is of lesser importance especially when the rate of interest used was on average higher than 6%.

7.2 Areas of Further Research

Investigate the effect of model uncertainty. Measuring model uncertainty is beyond the scope of this thesis. Two mortality projection models have been used in chapter 5 in order to illustrate the effect of choosing one model instead of the other but this is not sufficient as an investigation of the effect of model uncertainty. A full investigation of model uncertainty would postulate a reasonable number of possible models of both future mortality and interest rates, giving each a prior probability in the Bayesian framework.

Considering the Cohort effect. It could be of interest to extend the Bayesian analysis of annuity payments using the Lee – Carter (1992) projection

methodology to the modeling of age-period-cohort effect that proved to be of high significance for UK experience (Willets (2004)). Note that the extension of the Lee-Carter model to incorporate a cohort effect has been considered by Ranshaw and Haberman (2006).

Bayesian modelling of the entropy measure (H). This could help to better understand the way H behaves. For example, it would be useful to be able to answer questions such as: at a given level of interest rate, what would be the level of mortality that is considered too high or too low such that the cost of a life annuity is less likely to respond to any further changes in the force of mortality.

Appendix (1)

Using Incomplete gamma function to find values of H – Mortality follows the Gompertz law

Equation (4.15) can be written as

$$H = \frac{-\mu_x^b}{c-\alpha} + \frac{\mu_x^b}{c-\alpha} * \frac{I}{J} \quad (A1.1)$$

where

$$I = \int_0^t \exp\left(\frac{-\mu_x^b}{c-\alpha} ((\exp(c-\alpha)t) - 1)\right) \exp((c-\alpha-\delta)t) dt$$
$$J = \int_0^t \exp\left(\frac{-\mu_x^b}{c-\alpha} ((\exp(c-\alpha)t) - 1)\right) \exp(-\delta t) dt$$

Put

$$y = \exp[(c-\alpha)t] \Rightarrow \frac{dy}{dt} = (c-\alpha)y \Rightarrow dy = (c-\alpha)y dt$$

$$y^x = \exp[(c-\alpha-\delta)t] \Rightarrow x \log y = (c-\alpha-\delta)t \Rightarrow x = \frac{(c-\alpha-\delta)}{(c-\alpha)}$$

and $m = \exp(c-\alpha)s$

By changing variables I can be expressed as

$$\begin{aligned}
I &= \int_1^m y^{\frac{(c-\alpha-\delta)}{(c-\alpha)}} \exp\left(\frac{-\mu_x^b}{(c-\alpha)} y\right) \frac{dy}{y(c-\alpha)} \\
&= \frac{1}{c-\alpha} \int_1^m y^{\frac{-\delta}{(c-\alpha)}} \exp\left(\frac{-\mu_x^b}{(c-\alpha)} y\right) dy \\
&= \frac{1}{c-\alpha} \int_1^m y^{-\beta} \exp(-\sigma y) dy \quad (A1.2)
\end{aligned}$$

where

$$\begin{aligned}
\beta &= \frac{\delta}{c-\alpha} \\
\sigma &= \frac{\mu_x^b}{(c-\alpha)}
\end{aligned}$$

In the same manner a change of variables is also used for J , in this case y is defined as above but x is different such that

$$y^x = \exp[(-\delta)x] \Rightarrow x \log y = -\delta x \Rightarrow x = \frac{-\delta}{c-\alpha}$$

So J can be expressed as

$$\begin{aligned}
J &= \int_1^m y^{\frac{-\delta}{(c-\alpha)}} \exp\left(\frac{-\mu_x^b}{(c-\alpha)} y\right) \frac{dy}{y(c-\alpha)} \\
&= \frac{1}{c-\alpha} \int_1^m y^{\frac{-\delta}{(c-\alpha)}-1} \exp\left(\frac{-\mu_x^b}{(c-\alpha)} y\right) dy \\
&= \frac{1}{c-\alpha} \int_1^m y^{-\beta-1} \exp(-\sigma y) dy \quad (A1.3)
\end{aligned}$$

Where β and σ are as defined above.

From equations (A1.2) and (A1.3)

$$\begin{aligned}
 H &= \frac{-\mu_x^b}{c-\alpha} + \frac{\mu_x^b}{c-\alpha} * \frac{\frac{1}{c-\alpha} \int_1^m y^{-\beta} \exp(-\sigma y) dy}{\frac{1}{c-\alpha} \int_1^m y^{-\beta-1} \exp(-\sigma y) dy} \\
 &= \frac{-\mu_x^b}{c-\alpha} + \frac{\mu_x^b}{c-\alpha} * \frac{\int_1^m y^{-\beta} \exp(-\sigma y) dy}{\int_1^m y^{-\beta-1} \exp(-\sigma y) dy} \\
 &= \frac{-\mu_x^b}{c-\alpha} + \frac{\mu_x^b}{c-\alpha} * \frac{I_2}{J_2}
 \end{aligned} \tag{A1.4}$$

If we define Z such that

$$Z = \sigma y \Rightarrow \frac{dZ}{dy} = \sigma \Rightarrow \frac{dZ}{\sigma} = dy$$

By another change of variables I_2 can be expressed as

$$\begin{aligned}
 I_2 &= \int_1^m y^{-\beta} \exp(-\sigma y) dy \\
 &= \int_{\sigma}^{\sigma m} \left(\frac{Z}{\sigma} \right)^{-\beta} \exp(-Z) \frac{dZ}{\sigma} \\
 &= \sigma^{\beta+1} \int_{\sigma}^{\sigma m} Z^{-\beta} \exp(-Z) dZ
 \end{aligned} \tag{A1.5}$$

And J_2 can also be expressed as

$$\begin{aligned}
J_2 &= \int_1^m y^{-\beta-1} \exp(-\sigma y) dy \\
&= \int_{\sigma}^{\sigma m} \left(\frac{Z}{\sigma}\right)^{-\beta-1} \exp(-Z) \frac{dZ}{\sigma} \\
&= \sigma^{\beta+2} \int_{\sigma}^{\sigma m} Z^{-\beta-1} \exp(-Z) dZ
\end{aligned} \tag{A1.6}$$

So finally, using equations (A1.5) and (A1.6) the Entropy measure (H) in (A1.4) can be expressed as

$$\begin{aligned}
H &= \frac{-\mu_x^b}{c-\alpha} + \frac{\mu_x^b}{c-\alpha} * \frac{\sigma^{\beta+1} \int_{\sigma}^{\sigma m} Z^{-\beta} \exp(-Z) dZ}{\sigma^{\beta+2} \int_{\sigma}^{\sigma m} Z^{-\beta-1} \exp(-Z) dZ} \\
&= \frac{-\mu_x^b}{c-\alpha} + \frac{\mu_x^b}{c-\alpha} * \frac{\int_{\sigma}^{\sigma m} Z^{-\beta} \exp(-Z) dZ}{\sigma \int_{\sigma}^{\sigma m} Z^{-\beta-1} \exp(-Z) dZ} \\
&= \frac{-\mu_x^b}{c-\alpha} + \frac{\mu_x^b}{c-\alpha} * \frac{\left[\int_0^{\sigma m} Z^{-\beta} \exp(-Z) dZ - \int_0^{\sigma} Z^{-\beta} \exp(-Z) dZ \right]}{\sigma \left[\int_0^{\sigma m} Z^{-\beta-1} \exp(-Z) dZ - \int_0^{\sigma} Z^{-\beta-1} \exp(-Z) dZ \right]}
\end{aligned} \tag{A1.7}$$

It can be seen that, all the integrals in (A1.7) are of the form of a lower incomplete gamma function.

Appendix (2)

Partial derivatives of the entropy measure (H)

Using the Gompertz law assumption (section 4.4.1), the entropy measure H can be expressed as follows:

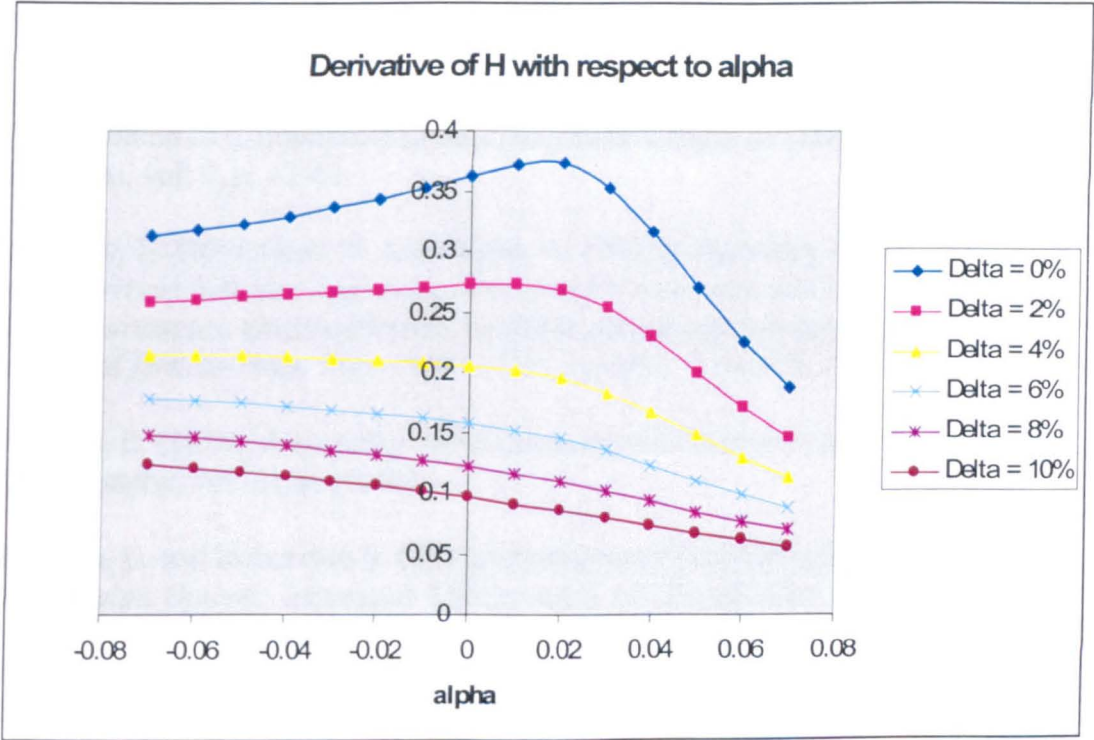
$$H = \frac{-\mu_x^b}{c-\alpha} + \frac{\mu_x^b}{c-\alpha} \left[\frac{\int_0^t \exp\left(\frac{-\mu_x^b}{c-\alpha}(\exp(c-\alpha)t)\right) \exp((c-\alpha-\delta)t) dt}{\int_0^t \exp\left(\frac{-\mu_x^b}{c-\alpha}(\exp(c-\alpha)t)\right) \exp(-\delta t) dt} \right] \quad (A2.1)$$

where b and c represent the base mortality table, α represents the level of mortality change and δ is the force of interest. Given that the value for μ_x^b is taken from the base table directly, hence it is a constant. Thus, $H(c, \alpha, \delta)$ is a function of α , c and δ .

The partial derivative of H with respect to α

$$\begin{aligned}
 \frac{\partial H}{\partial \alpha} = & \frac{-\mu_x^b}{(c-\alpha)^2} + \frac{\mu_x^b}{c-\alpha} \left[\frac{\int_0^t \exp\left(\frac{-\mu_x^b}{c-\alpha}(\exp(c-\alpha)t)\right) \exp((c-\alpha-\delta)t) dt}{\int_0^t \exp\left(\frac{-\mu_x^b}{c-\alpha}(\exp(c-\alpha)t)\right) \exp(-\delta t) dt} \right] \\
 & + \frac{\mu_x^b}{c-\alpha} \left[\frac{\int_0^t \left(\exp\left(\frac{-\mu_x^b}{c-\alpha}(\exp(c-\alpha)t) + (c-\alpha-\delta)t\right) \right) \left(-t - \frac{\mu_x^b}{c-\alpha}(\exp(c-\alpha)t) + t \frac{\mu_x^b}{c-\alpha}(\exp(c-\alpha)t) \right) dt}{\int_0^t \exp\left(\frac{-\mu_x^b}{c-\alpha}(\exp(c-\alpha)t)\right) \exp(-\delta t) dt} \right] \\
 & - \frac{\mu_x^b}{c-\alpha} \left[\frac{\int_0^t \exp\left(\frac{-\mu_x^b}{c-\alpha}(\exp(c-\alpha)t) + (c-\alpha-\delta)t\right) dt \int_0^t \left(-\frac{\mu_x^b}{c-\alpha}(\exp(c-\alpha)t) + t \frac{\mu_x^b}{c-\alpha}(\exp(c-\alpha)t) \right) \exp\left(\frac{-\mu_x^b}{c-\alpha}(\exp(c-\alpha)t)\right) \exp(-\delta t) dt}{\left(\int_0^t \exp\left(\frac{-\mu_x^b}{c-\alpha}(\exp(c-\alpha)t)\right) \exp(-\delta t) dt \right)^2} \right] \quad (A2.2)
 \end{aligned}$$

Figure A2.1: The partial derivative of H with respect to α at different interest rates



Bibliography and References

Alho, J.M. (2000), *Discussion of Lee (2000)*. North American Actuarial Journal, vol. 4, p. 91–93.

Andreev, E.M. (1982), *Method Komponent v Analize Prodoljitelnosty Zjizni*. [The Method of Components in the Analysis of Length of Life]. Vestnik Statistiki, vol. 9, p. 42-47.

Andreev, E. Shkolnikov, V. and Begun, A. (2002), *Algorithm for decomposition of differences between aggregate demographic measures and its application to life expectancies, Ginicoefficients, health expectancies, parity-progression ratios and total fertility rates*. Submitted to Demographic Research.

Arriaga, E. (1984), *Measuring and Explaining the Change in Life Expectancies*. Demography, vol. 21, p. 83-96.

Ballotta, L. and Haberman S. (2003), *Valuation of Guaranteed Annuity Conversion Options*. Insurance: Mathematics and Economics, vol.33, p. 87-108.

Benjamin, B. (1964), *Demographic and Actuarial Aspects of Ageing, with Special Reference to England and Wales*. J.I.A., vol. 90, p. 211-253.

Benjamin, B. (1982), *The Span of Life*. J.I.A., vol. 109, p. 319-340.

Benjamin, B. & Pollard, J. H. (1993), *The Analysis of Mortality and other Actuarial Statistics*, 3rd Edition. Institute of Actuaries and Faculty of Actuaries.

Benjamin, B and Soliman, A. (1993), *Mortality on the Move*. Distributed by Actuarial Education Service, printed by City University London.

Björk, T. (1998), *Arbitrage Theory in Continuous Time*. Oxford University Press.

Black, F. Derman, E. and Toy, W. (1990), *A One-Factor Model of Interest Rates and Its Application to Treasury Bond Options*. Financial Analysts Journal, vol.46, p.33-39.

Black, F. and Karasinski, P. (1991), *Bond and Option Pricing When Short Rates Are Lognormal*. Financial Analysts Journal vol. 47, p. 52-59.

Bourgeois-Pichat, J. (1952), *Essai sur la mortalité 'biologique' de l'homme*. Population, vol. 3, p. 381-394.

- Box, G. and Taio, G. (1973), *Bayesian inference in statistical analysis*. Addison-Wesley Series in Behavioral Science: Quantitative methods.
- Brace, A. Gatarek and D. Musiela, M. (1997), *The Market Model of Interest Rate Dynamics*. Mathematical Finance, vol. 7, p. 127–155.
- Brass, W. (1971), *On the Scale of Mortality* in Brass, W., ed., *Biological Aspects of Demography*. Taylor and Francis, London.
- Brass, W. (1974), *Mortality models and their uses in demography*. Transactions of the faculty of actuaries, vol. 33, p. 123-132.
- Brigo, D. and Mercurio, F. (2001), *Interest Rate Models: Theory and Practice*. Springer.
- Brillinger, D.R. (1986), *The natural variability of vital rates and associated statistics*. Biometrics, vol. 42, p.693–734.
- Brouhns, N., Denuit, M. and Vermunt, J. K. (2002a), *A Poisson Log-bilinear Regression Approach to the Construction of Projected Lifetables*. Insurance: Mathematics and Economics, vol. 31, p. 373-393.
- Brouhns, N., Denuit, M. and Vermunt, J. K. (2002b), *Measuring the longevity risk in mortality projections*. Bulletin of the Swiss Association of Actuaries 2, pp. 105-130.
- Cairns, A.J.G. (2000), *A discussion of parameter and model uncertainty in insurance*. Insurance: Mathematics and Economics, vol. 27, p. 313–330.
- Carlin, B. P. (1992), *A Simple Monte Carlo Approach to Bayesian Graduation*. Transactions of the Society of Actuaries, vol. 44, p. 55-76.
- Carriere, J. F. (1994), *Dependent Decrement Theory*. Transactions of the Society of Actuaries, vol. 46, p. 45-74.
- Charlton, J. (1997), *Trends in all-cause mortality, 1841-1994. The Health of Adult Britain, 1841-1994*. Office for national statistics, vol. 1, Decennial Supplement No. 12. Chapter 4. HMSO, London
- Chiarella, C. and Kwon, O. (2001), *Forward rate dependent Markovian transformations of the health-jarrow morton term structure model*. Finance and Stochastics, vol.5:2, p. 237-258.

- Continuous Mortality Investigation Bureau (1998) *Continuous Mortality Investigation Reports*. CMIR, 16, The Institute of Actuaries and the Faculty of Actuaries, United Kingdom.
- CMI (2002), *Working Paper 1: An interim basis for adjusting the 92" Series mortality projections for cohort effects*. The Faculty of Actuaries and Institute of Actuaries.
- CMI (2004), *Working Paper 3: Projecting future mortality: A discussion paper*. The Faculty of Actuaries and Institute of Actuaries.
- CMI (2005c), *Working Paper 15: Projecting future mortality: Towards a proposal for a stochastic methodology*. The Faculty of Actuaries and Institute of Actuaries.
- Coale, A. J. & Demeny, P. (1966), *Regional Model Life Tables and Stable Populations*. Princeton University Press, Princeton.
- Congdon, P. (1993), *Statistical Graduation in Local Demographic Analysis and Projection*, J. R. Statistic. Soc. A, vol. 156, p. 237-270.
- Cowles, M.K. and Carlin, B.P.(1996), *Markov Chain Monte Carlo Convergence Diagnostics: A comparative Review*. Journal of the American statistical association, Vol. 91, p. 883-904.
- Cox, J.C., Ingersoll, J.E. and Ross, S.A. (1985), *A theory of the Term Structure of Interest Rates*. Econometrica, 53, 2, p. 385-407.
- Cramer, H. and Wold, H. (1935), *Mortality variations in Sweden: a study in graduation and forecasting*. Skandinavisk Aktuarieidskrift, vol. 18, p. 161--241
- Currie, I.D., Durban, M. and Eilers, P.H.C. (2004), *Smoothing and forecasting mortality rates*. Statistical Modelling, vol. 4, p. 279-298.
- Czado, C., Delwarde, A., Denuit, M. (2005), *Bayesian Poisson log-bilinear mortality projections*. Insurance: Mathematics and Economics, vol. 36, p. 260–284.
- Demetrius, L (1976), *Measures of variability in age-structured populations*. Journal of Theoretical Biology, vol. 63, p. 397-404.
- Devolder, P. (1986), *OpErations Stochastiques de Capitalisation*. ASTIN Bulletin 16S, p.S5-S30.

- Dufresne, D. (1990), *The Distribution of a Perpetuity, with Applications to Risk Theory and Pension Funding*. Scandinavian Actuarial Journal, p.39-79.
- Finkelstein, A. and Poteba, J. (2002), *Selection Effects in the United Kingdom Individual Annuity Market*. The Economic Journal, vol. 112, p. 28-50.
- Forfar, D. O., McCutcheon, M. A. & Wilkie, A. D. (1988), *On Graduation By Mathematical Formula*. J.I.A., vol. 115, p. 1-149.
- Forfar, D. and Smith, D. (1988), *The Changing Shape of English Life Tables*. Transactions of the faculty of actuaries, vol. 40, p. 98-134.
- Frees, E.W. (1990), *Stochastic Life Contingencies with solvency considerations*. Transactions of the Society of Actuaries, vol. XLII, p. 91-148.
- Gelfand, A.E., Hills, S.E., Racine-Poon, A. and Smith, A.F.M. (1990), *Illustration of Bayesian Inference in Normal Data Models Using Gibbs Sampling*. Journal of the American Statistical Association, vol. 85, p. 972-85.
- Gelfand, A.E. and Smith, A.F.M. (1990), *Sampling-based Approximations to calculating Marginal Densities*. Journal of the American Statistical Association, vol. 85, p. 398-409.
- Gelman, A. and Rubin, D. (1992), *Inference from Iterative Simulation Using Multiple Sequences*. Statistical Science, vol. 7, p. 457-511.
- Gelman, A. Carlin, J.B Stern, H. S. and Rubin, D. (1995), *Bayesian Data Analysis*, London:Chapman & Hall.
- Geman, S. and Geman, D. (1984), *Stochastic Relaxation, Gibbs Distributions, and the Bayesian Restoration of Images*. IEEE Transactions on Pattern Analysis and Machine Intelligence, vol. 6, p. 721-41.
- Gilks, W. R., Richardson, S. and Spiegelhalter, D.J. (1996), *Markov Chain Monte Carlo Methods in Practice*. Chapman and Hall
- Girosi, F. and King, G. (2005), *A Reassessment of the Lee-Carter Mortality Forecasting Method*. <http://gking.harvard.edu/files/abs/lc-abs.shtml>.
- Goldman, N. and Lord, G. (1986), *A New Look at Entropy and the Life Table*, Demography, Vol 23, pp. 275-282.
- Goldman, N. and Lord, G. (1987), Reply to "A New Look at Entropy and the Life Table" by S.Mtra, Demography, Vol 24, pp. 441-442.

- Gompertz, B. (1825), *On the Nature of the Function of the Law of Human Mortality and on a New Mode of Determining the Value of Life Contingencies*. Phil. Trans. of Royal Society, vol. 115, p. 513-585.
- Hakkert, R. (1987), *Life Table Transformations and Inequality Measures: Some Noteworthy Formal Relationships*. Demography, Vol 4, pp. 615-622.
- Hastings, W.K. (1970), *Monte Carlo Sampling Methods Using Markov Chains and Their Applications*. Biometrika, vol. 57, p. 97-109.
- Heath D., Jarrow R. and Morton K. (1990), *Bond Pricing and the Term Structure of Interest Rates, A discrete Time Approximation*. Journal of Financial and Quantitative Analysis, vol. 25, p.419-440.
- Heath D., Jarrow R. and Morton K. (1992), *Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation*. Econometrica, Vol. 60, p. 77-105.
- Heligman, L. & Pollard, J. H. (1980), *The Age Pattern of Mortality*. J.I.A., vol. 107, p. 49-80.
- Ho T.S.Y. and Lee S.B. (1986), *Term Structure Movements and Pricing Interest Rate Contingent Claims*. Journal of Finance, vol. 41, p.1011-1029.
- Hull, J. and White, A. (1990), *Pricing Interest Rate Derivatives Securities*. The Review of Financial Studies, vol. 3, no. 4, p. 573-92.
- Jamshidian, F. (1997), *LIBOR and swap market models and measures*. Finance and Stochastics, vol. 1, p. 293-330.
- Jarrow, R. and Turnbull, SM. (1994), *Delta, Gamma and Bucket Hedging of Interest Rate Derivatives*. Applied Mathematical Finance, vol. 1, p. 21-48.
- Keyfitz, N. (1977, 1985), *Applied Mathematical Demography*. Second Edition, Springer-Verlag, New York.
- Keyfitz, N. and Flieger, W. (1968), *World Population: An analysis of vital data*. Chicago: university of Chicago Press.
- Khalaf-Allah, M., Haberman, S. and Verrall, R. (2006), *Measuring the effect of mortality improvements on the cost of annuities*. Insurance: Mathematics and Economics, vol. 39, p. 231-249.

- Khinchin, A.I. (1957), *Mathematical Foundations of Information Theory*. Dover, New York .
- Lee R. (2000), *The Lee-Carter method for forecasting mortality, with various extensions and applications*. North American Actuarial Journal, vol. 4, p. 80-93.
- Lee R. and Carter L. (1992), *Modelling and Forecasting US Mortality*. Journal of American Statistical Association, vol. 87, p 659-671.
- Leser, C. (1955), *Variation in Mortality and Life Expectation*. Population Studies, vol. 9, pp. 67-71.
- MacDonald A.S., Cairns, A.J.G., Gwilt, P.L. and Miller, K.A. (1998), *An International Comparison of Recent Trends in Population Mortality*. British Actuarial Journal, vol. 4, p. 3-141.
- Makeham, W. M. (1860), *On the Law of Mortality*. J.I.A., vol. 13, p. 325-358.
- McCrory, R.T (1986), *Mortality Risk in Life Annuities*. Transactions of the Society of Actuaries, vol. 36, p.309-338.
- Metropolis, N., Rosenbluth, A.W., Rosenbluth, M.N. and Teller E. (1953), *Equations of State Calculations by Fast Computing Machines*. Journal of Chemical Physics, vol. 21, p. 1087-92.
- Norberg R. (1993), *A Solvency Study In Life Insurance*. Proceedings of the 3rd AFIR International Colloquium, Rome. p. 822-830.
- Nordenmark, N.V.E. (1906), *Über die Bedeutung der Verlängerung der Lebensdauer für die Berechnung der Leibrenten*. Transactions of the fifth International congress of Actuaries, Vol. 1, Berlin, p. 421-430.
- Olivieri A. (2001), *Uncertainty in mortality projections: an actuarial perspective*. Insurance: Mathematics & Economics, vol. 29, p. 231-245.
- Olivieri, A. and Pitaco, E. (2002), *Inference about Mortality improvement in Life Annuity Portfolios*. Presented to 27th international Congress of Actuaries, Cancun, Mexico.
- Organisation for Economic Co-operation and Development (1980), *Mortality in Developing Countries*, vol. III. *New Model Life Tables for Use in Developing Countries*. Paris.

- Panjer H.H. and Bellhouse D.R. (1980), *Stochastic Modelling of Interest Rates with Applications to Life Contingencies*. Journal of Risk and Insurance, vol. 47, p.91-110.
- Panjer H.H. and Bellhouse D.R. (1981), *Stochastic Modelling of Interest Rates with Applications to Life Contingencies Part II*. Journal of Risk and Insurance, vol. 48, p.628-637.
- Parker G. (1994a), *Moments of the present value of a portfolio of policies*. Scandinavian Actuarial Journal, 1994; 1, p.53-67.
- Parker G. (1994b), *Limiting distribution of the present value of a portfolio*. ASTIN Bulletin, vol. 24, p.47-60.
- Perks, W. (1932), *On Some Experiments in Graduation of Mortality Statistics*. J.I.A., vol. 63, p.12-40.
- Pitacco, E. (2004), *Survival models in a dynamic context: a survey*. Insurance: Mathematics & Economics, Vol. 35, p. 279–298.
- Pollard, A.H. (1949), *Methods of Forecasting Mortality Using Australian Data*. J.I.A., vol. 75, p. 151-170.
- Pollard, J.H (1982), *The Expectation of Life And Its Relationship to Mortality*. Journal of the Institute of Actuaries, vol. 109, p.225-240.
- Pollard, J. H. (1987), *Projection of Age-Specific Mortality Rates*. Population Bulletin of the United Nations 21, 22, p. 55-69.
- Pollard, J. H. (1988), *On the Decomposition of Changes in Expectation of Life and Differentials in life Expectancy*. Demography, vol. 25, p. 265-276.
- Pressat, R. (1985), *Contribution des Écarts de Mortalité par Âge à la Différence des Vies Moyennes*. Population, 4-5, p. 765-770.
- Renshaw, A.E. and Haberman, S. (2003a), *Lee-Carter Mortality Forecasting, a Parallel GLM Approach, England & Wales Mortality Projections*. Journal of Royal Statistical Society, C (Applied Statistics), vol. 52, p. 119-137.
- Renshaw, A.E. and Haberman, S. (2003b), *On the Forecasting of Mortality Reduction Factors*. Insurance: Mathematics and Economics, vol. 32, p. 379–401.
- Renshaw, A., Haberman, S. and Hatzopoulos, P. (1996), *Modelling of Recent Mortality Trends in UK Male Assured Lives*. British Actuarial Journal, vol. 2, p. 449-477.

- Sandmann, K. and Sondermann, D. (1993), *A Term Structure Model and the Pricing of Interest Rate Derivatives*. Review of Futures Markets, 12, p. 391-423.
- Schoutens, W. (2003), *Lévy processes in finance: pricing financial derivatives*. Wiley series in probability and statistics.
- Scollnik, D.P.M. (2001), *Actuarial Modelling with MCMC and BUGS*. North American Actuarial Journal, vol. 5, pp 96-124.
- Shannon, C.E. (1948), *A Mathematical Theory of Communication*. Bell System Technical Journal, vol. 27, p. 379-423.
- Sithole, T.Z., Haberman, S. and Verrall, R.J. (2000). *An Investigation into Parametric Models For Mortality Projections, With Applications to Immediate Annuitants' and Life Office Pensioners' Data*. Insurance: Mathematics and Economics, vol.27, p.285-312.
- Thatcher, A.R. (1990), *Some Results on the Gompertz and Heligman Laws of Mortality*. J.I.A., vol. 117, p. 135-151.
- Thatcher, A.R. (1999), *The Long Term Pattern of Adult Mortality and Highest Attended Age*. Journal of Royal Statistical Society, Series A, vol. 162, p 5-43.
- Thiele, P. (1872), *On a Mathematical Formula to Express the Rate of Mortality Throughout the Whole of Life*.
- Vasicek O.A. (1977), *An Equilibrium Characterization of the Term Structure*. Journal of Financial Economics, vol. 5, p.177-188.
- Waters H.R. (1978), *The Moments and Distributions of Actuarial Functions*. Journal of the Institute of Actuaries, vol. 105, Part 1, p. 61-75.
- Waters, H.R and Wilkie A.D. (1987), *A Short Note On The Construction Of Life Tables And Multiple Decrement Tables*. Journal of the Institute of Actuaries, vol. 114, p. 569-580.
- Willeits, R.C., Gallop, A.P., Leandro, P.A., Lu, J.L.C., Macdonald, A.S., Miller, K.A., Richards, S.J., Robjohns, N., Ryan, J.P. & Waters, H.R. (2004), *Longevity in the 21st Century*. British Actuarial Journal, vol. 10, p. 685-832.

Willets, R.C. (2004), *The cohort effect: Insights and explanations*. British Actuarial Journal, vol. 10, p. 833-877.

Wong-Fupuy, C., Haberman, S. (2004), *Projecting mortality trends: Recent developments in the United kingdom and the United States*. North American Actuarial Journal, vol. 8, p. 56-83.

Zellner, A. (1971), *An Introduction to Bayesian Inference in Econometrics*. New York: John Wiley.